

# 10 | FIXED-AXIS ROTATION



**Figure 10.1** Brazos wind farm in west Texas. As of 2012, wind farms in the US had a power output of 60 gigawatts, enough capacity to power 15 million homes for a year. (credit: modification of work by “ENERGY.GOV”/Flickr)

## Chapter Outline

- 10.1 Rotational Variables
- 10.2 Rotation with Constant Angular Acceleration
- 10.3 Relating Angular and Translational Quantities
- 10.4 Moment of Inertia and Rotational Kinetic Energy
- 10.5 Calculating Moments of Inertia
- 10.6 Torque
- 10.7 Newton’s Second Law for Rotation
- 10.8 Work and Power for Rotational Motion

## Introduction

In previous chapters, we described motion (kinematics) and how to change motion (dynamics), and we defined important concepts such as energy for objects that can be considered as point masses. Point masses, by definition, have no shape and so can only undergo translational motion. However, we know from everyday life that rotational motion is also very important and that many objects that move have both translation and rotation. The wind turbines in our chapter opening image are a prime example of how rotational motion impacts our daily lives, as the market for clean energy sources continues to grow.

We begin to address rotational motion in this chapter, starting with fixed-axis rotation. Fixed-axis rotation describes the rotation around a fixed axis of a rigid body; that is, an object that does not deform as it moves. We will show how to apply all the ideas we’ve developed up to this point about translational motion to an object rotating around a fixed axis. In the next chapter, we extend these ideas to more complex rotational motion, including objects that both rotate and translate, and objects that do not have a fixed rotational axis.

## 10.1 | Rotational Variables

### Learning Objectives

By the end of this section, you will be able to:

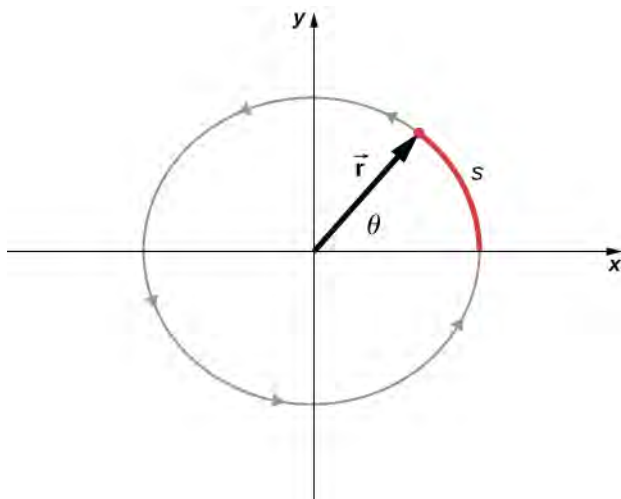
- Describe the physical meaning of rotational variables as applied to fixed-axis rotation
- Explain how angular velocity is related to tangential speed
- Calculate the instantaneous angular velocity given the angular position function
- Find the angular velocity and angular acceleration in a rotating system
- Calculate the average angular acceleration when the angular velocity is changing
- Calculate the instantaneous angular acceleration given the angular velocity function

So far in this text, we have mainly studied translational motion, including the variables that describe it: displacement, velocity, and acceleration. Now we expand our description of motion to rotation—specifically, rotational motion about a fixed axis. We will find that rotational motion is described by a set of related variables similar to those we used in translational motion.

### Angular Velocity

Uniform circular motion (discussed previously in **Motion in Two and Three Dimensions**) is motion in a circle at constant speed. Although this is the simplest case of rotational motion, it is very useful for many situations, and we use it here to introduce rotational variables.

In **Figure 10.2**, we show a particle moving in a circle. The coordinate system is fixed and serves as a frame of reference to define the particle's position. Its position vector from the origin of the circle to the particle sweeps out the angle  $\theta$ , which increases in the counterclockwise direction as the particle moves along its circular path. The angle  $\theta$  is called the **angular position** of the particle. As the particle moves in its circular path, it also traces an arc length  $s$ .



**Figure 10.2** A particle follows a circular path. As it moves counterclockwise, it sweeps out a positive angle  $\theta$  with respect to the  $x$ -axis and traces out an arc length  $s$ .

The angle is related to the radius of the circle and the arc length by

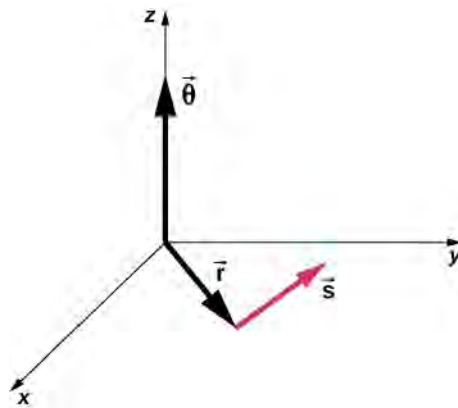
$$\theta = \frac{s}{r}. \quad (10.1)$$

The angle  $\theta$ , the angular position of the particle along its path, has units of radians (rad). There are  $2\pi$  radians in  $360^\circ$ . Note that the radian measure is a ratio of length measurements, and therefore is a dimensionless quantity. As the particle moves along its circular path, its angular position changes and it undergoes angular displacements  $\Delta\theta$ .

We can assign vectors to the quantities in **Equation 10.1**. The angle  $\vec{\theta}$  is a vector out of the page in **Figure 10.2**. The angular position vector  $\vec{r}$  and the arc length  $\vec{s}$  both lie in the plane of the page. These three vectors are related to each other by

$$\vec{s} = \vec{\theta} \times \vec{r}. \quad (10.2)$$

That is, the arc length is the cross product of the angle vector and the position vector, as shown in **Figure 10.3**.



**Figure 10.3** The angle vector points along the  $z$ -axis and the position vector and arc length vector both lie in the  $xy$ -plane. We see that  $\vec{s} = \vec{\theta} \times \vec{r}$ . All three vectors are perpendicular to each other.

The magnitude of the **angular velocity**, denoted by  $\omega$ , is the time rate of change of the angle  $\theta$  as the particle moves in its circular path. The **instantaneous angular velocity** is defined as the limit in which  $\Delta t \rightarrow 0$  in the average angular velocity  $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$ :

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}, \quad (10.3)$$

where  $\theta$  is the angle of rotation (**Figure 10.2**). The units of angular velocity are radians per second (rad/s). Angular velocity can also be referred to as the rotation rate in radians per second. In many situations, we are given the rotation rate in revolutions/s or cycles/s. To find the angular velocity, we must multiply revolutions/s by  $2\pi$ , since there are  $2\pi$  radians in one complete revolution. Since the direction of a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

We can see how angular velocity is related to the tangential speed of the particle by differentiating **Equation 10.1** with respect to time. We rewrite **Equation 10.1** as

$$s = r\theta.$$

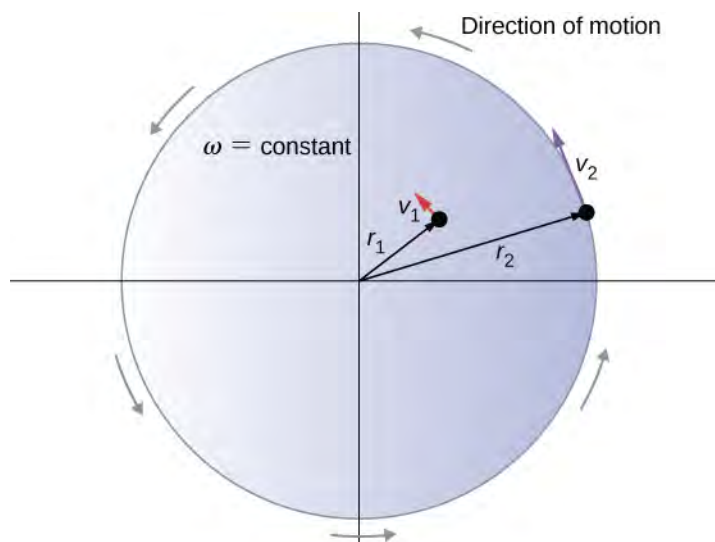
Taking the derivative with respect to time and noting that the radius  $r$  is a constant, we have

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = \theta \frac{dr}{dt} + r \frac{d\theta}{dt} = r \frac{d\theta}{dt}$$

where  $\theta \frac{dr}{dt} = 0$ . Here  $\frac{ds}{dt}$  is just the tangential speed  $v_t$  of the particle in **Figure 10.2**. Thus, by using **Equation 10.3**, we arrive at

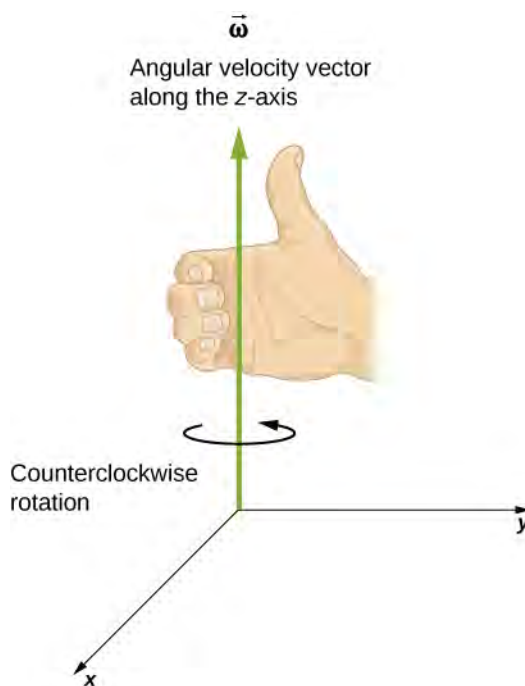
$$v_t = r\omega. \quad (10.4)$$

That is, the tangential speed of the particle is its angular velocity times the radius of the circle. From **Equation 10.4**, we see that the tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. This effect is shown in **Figure 10.4**. Two particles are placed at different radii on a rotating disk with a constant angular velocity. As the disk rotates, the tangential speed increases linearly with the radius from the axis of rotation. In **Figure 10.4**, we see that  $v_1 = r_1\omega_1$  and  $v_2 = r_2\omega_2$ . But the disk has a constant angular velocity, so  $\omega_1 = \omega_2$ . This means  $\frac{v_1}{r_1} = \frac{v_2}{r_2}$  or  $v_2 = \left(\frac{r_2}{r_1}\right)v_1$ . Thus, since  $r_2 > r_1$ ,  $v_2 > v_1$ .



**Figure 10.4** Two particles on a rotating disk have different tangential speeds, depending on their distance to the axis of rotation.

Up until now, we have discussed the magnitude of the angular velocity  $\omega = d\theta/dt$ , which is a scalar quantity—the change in angular position with respect to time. The vector  $\vec{\omega}$  is the vector associated with the angular velocity and points along the axis of rotation. This is useful because when a rigid body is rotating, we want to know both the axis of rotation and the direction that the body is rotating about the axis, clockwise or counterclockwise. The angular velocity  $\vec{\omega}$  gives us this information. The angular velocity  $\vec{\omega}$  has a direction determined by what is called the right-hand rule. The right-hand rule is such that if the fingers of your right hand wrap counterclockwise from the  $x$ -axis (the direction in which  $\theta$  increases) toward the  $y$ -axis, your thumb points in the direction of the positive  $z$ -axis (**Figure 10.5**). An angular velocity  $\vec{\omega}$  that points along the positive  $z$ -axis therefore corresponds to a counterclockwise rotation, whereas an angular velocity  $\vec{\omega}$  that points along the negative  $z$ -axis corresponds to a clockwise rotation.



**Figure 10.5** For counterclockwise rotation in the coordinate system shown, the angular velocity points in the positive z-direction by the right-hand-rule.

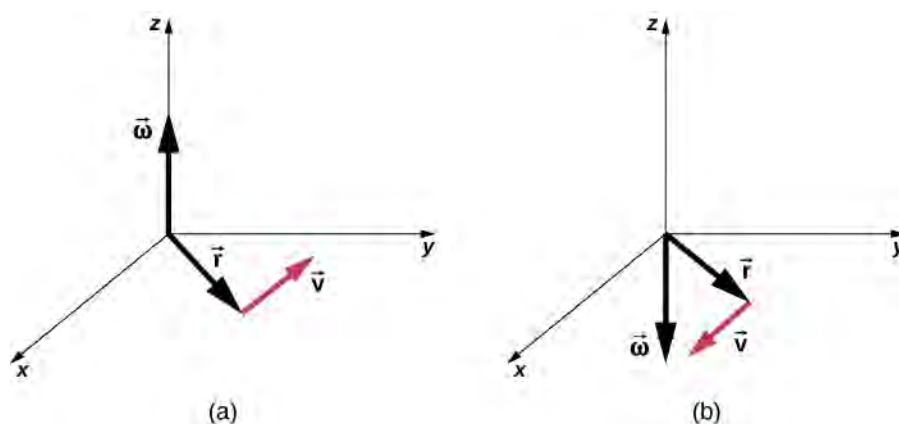
We can verify the right-hand-rule using the vector expression for the arc length  $\vec{s} = \vec{\theta} \times \vec{r}$ , **Equation 10.2**. If we differentiate this equation with respect to time, we find

$$\frac{d\vec{s}}{dt} = \frac{d}{dt}(\vec{\theta} \times \vec{r}) = \left( \frac{d\vec{\theta}}{dt} \times \vec{r} \right) + \left( \vec{\theta} \times \frac{d\vec{r}}{dt} \right) = \frac{d\vec{\theta}}{dt} \times \vec{r}.$$

Since  $\vec{r}$  is constant, the term  $\vec{\theta} \times \frac{d\vec{r}}{dt} = 0$ . Since  $\vec{v} = \frac{d\vec{s}}{dt}$  is the tangential velocity and  $\vec{\omega} = \frac{d\vec{\theta}}{dt}$  is the angular velocity, we have

$$\vec{v} = \vec{\omega} \times \vec{r}. \quad (10.5)$$

That is, the tangential velocity is the cross product of the angular velocity and the position vector, as shown in **Figure 10.6**. From part (a) of this figure, we see that with the angular velocity in the positive z-direction, the rotation in the xy-plane is counterclockwise. In part (b), the angular velocity is in the negative z-direction, giving a clockwise rotation in the xy-plane.



**Figure 10.6** The vectors shown are the angular velocity, position, and tangential velocity. (a) The angular velocity points in the positive  $z$ -direction, giving a counterclockwise rotation in the  $xy$ -plane. (b) The angular velocity points in the negative  $z$ -direction, giving a clockwise rotation.

## Example 10.1

### Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of  $\theta = \omega t = (45.0 \text{ rad/s})t$  radians. The wheel rotates counterclockwise when viewed in the plane of the page. (a) What is the angular velocity of the flywheel? (b) What direction is the angular velocity? (c) How many radians does the flywheel rotate through in 30 s? (d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation?

### Strategy

The functional form of the angular position of the flywheel is given in the problem as  $\theta(t) = \omega t$ , so by taking the derivative with respect to time, we can find the angular velocity. We use the right-hand rule to find the angular velocity. To find the angular displacement of the flywheel during 30 s, we seek the angular displacement  $\Delta\theta$ , where the change in angular position is between 0 and 30 s. To find the tangential speed of a point at a distance from the axis of rotation, we multiply its distance times the angular velocity of the flywheel.

### Solution

- $\omega = \frac{d\theta}{dt} = 45 \text{ rad/s}$ . We see that the angular velocity is a constant.
- By the right-hand rule, we curl the fingers in the direction of rotation, which is counterclockwise in the plane of the page, and the thumb points in the direction of the angular velocity, which is out of the page.
- $\Delta\theta = \theta(30 \text{ s}) - \theta(0 \text{ s}) = 45.0(30.0 \text{ s}) - 45.0(0 \text{ s}) = 1350.0 \text{ rad}$ .
- $v_t = r\omega = (0.1 \text{ m})(45.0 \text{ rad/s}) = 4.5 \text{ m/s}$ .

### Significance

In 30 s, the flywheel has rotated through quite a number of revolutions, about 215 if we divide the angular displacement by  $2\pi$ . A massive flywheel can be used to store energy in this way, if the losses due to friction are minimal. Recent research has considered superconducting bearings on which the flywheel rests, with zero energy loss due to friction.

## Angular Acceleration

We have just discussed angular velocity for uniform circular motion, but not all motion is uniform. Envision an ice skater spinning with his arms outstretched—when he pulls his arms inward, his angular velocity increases. Or think about a computer's hard disk slowing to a halt as the angular velocity decreases. We will explore these situations later, but we can already see a need to define an **angular acceleration** for describing situations where  $\omega$  changes. The faster the change in  $\omega$ , the greater the angular acceleration. We define the **instantaneous angular acceleration**  $\alpha$  as the derivative of angular

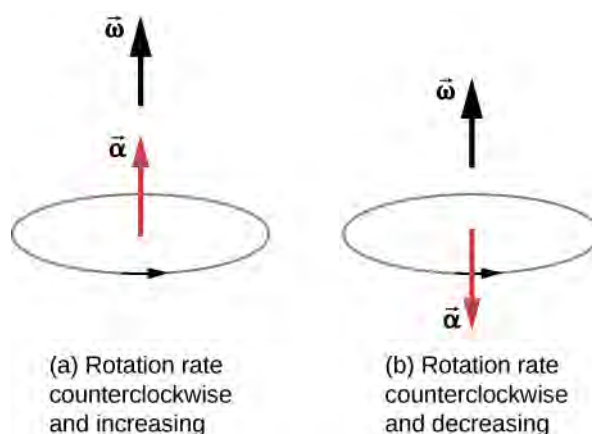
velocity with respect to time:

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \quad (10.6)$$

where we have taken the limit of the average angular acceleration,  $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$  as  $\Delta t \rightarrow 0$ .

The units of angular acceleration are (rad/s)/s, or  $\text{rad/s}^2$ .

In the same way as we defined the vector associated with angular velocity  $\vec{\omega}$ , we can define  $\vec{\alpha}$ , the vector associated with angular acceleration (Figure 10.7). If the angular velocity is along the positive z-axis, as in Figure 10.5, and  $\frac{d\omega}{dt}$  is positive, then the angular acceleration  $\vec{\alpha}$  is positive and points along the +z- axis. Similarly, if the angular velocity  $\vec{\omega}$  is along the positive z-axis and  $\frac{d\omega}{dt}$  is negative, then the angular acceleration is negative and points along the -z- axis.

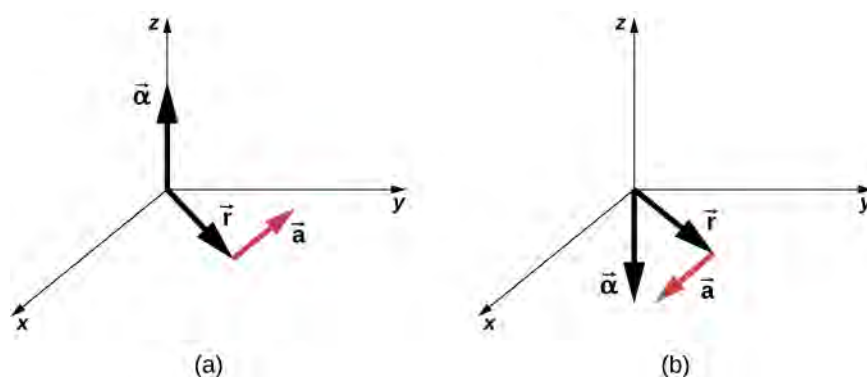


**Figure 10.7** The rotation is counterclockwise in both (a) and (b) with the angular velocity in the same direction. (a) The angular acceleration is in the same direction as the angular velocity, which increases the rotation rate. (b) The angular acceleration is in the opposite direction to the angular velocity, which decreases the rotation rate.

We can express the tangential acceleration vector as a cross product of the angular acceleration and the position vector. This expression can be found by taking the time derivative of  $\vec{v} = \vec{\omega} \times \vec{r}$  and is left as an exercise:

$$\vec{a} = \vec{\alpha} \times \vec{r}. \quad (10.7)$$

The vector relationships for the angular acceleration and tangential acceleration are shown in Figure 10.8.



**Figure 10.8** (a) The angular acceleration is the positive  $z$ -direction and produces a tangential acceleration in a counterclockwise sense. (b) The angular acceleration is in the negative  $z$ -direction and produces a tangential acceleration in the clockwise sense.

We can relate the tangential acceleration of a point on a rotating body at a distance from the axis of rotation in the same way that we related the tangential speed to the angular velocity. If we differentiate [Equation 10.4](#) with respect to time, noting that the radius  $r$  is constant, we obtain

$$a_t = r\alpha. \quad (10.8)$$

Thus, the tangential acceleration  $a_t$  is the radius times the angular acceleration. [Equation 10.4](#) and [Equation 10.8](#) are important for the discussion of rolling motion (see [Angular Momentum](#)).

Let's apply these ideas to the analysis of a few simple fixed-axis rotation scenarios. Before doing so, we present a problem-solving strategy that can be applied to rotational kinematics: the description of rotational motion.

### Problem-Solving Strategy: Rotational Kinematics

1. Examine the situation to determine that rotational kinematics (rotational motion) is involved.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A sketch of the situation is useful.
3. Make a complete list of what is given or can be inferred from the problem as stated (identify the knowns).
4. Solve the appropriate equation or equations for the quantity to be determined (the unknown). It can be useful to think in terms of a translational analog, because by now you are familiar with the equations of translational motion.
5. Substitute the known values along with their units into the appropriate equation and obtain numerical solutions complete with units. Be sure to use units of radians for angles.
6. Check your answer to see if it is reasonable: Does your answer make sense?

Now let's apply this problem-solving strategy to a few specific examples.

## Example 10.2

### A Spinning Bicycle Wheel

A bicycle mechanic mounts a bicycle on the repair stand and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the average angular acceleration in  $\text{rad/s}^2$ . (b) If she now hits the brakes, causing an angular acceleration of  $-87.3 \text{ rad/s}^2$ , how long does it take the wheel to stop?



### Strategy

The average angular acceleration can be found directly from its definition  $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$  because the final angular velocity and time are given. We see that  $\Delta\omega = \omega_{\text{final}} - \omega_{\text{initial}} = 250 \text{ rev/min}$  and  $\Delta t$  is 5.00 s. For part (b), we know the angular acceleration and the initial angular velocity. We can find the stopping time by using the definition of average angular acceleration and solving for  $\Delta t$ , yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}.$$

### Solution

- a. Entering known information into the definition of angular acceleration, we get

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{250 \text{ rpm}}{5.00 \text{ s}}.$$

Because  $\Delta\omega$  is in revolutions per minute (rpm) and we want the standard units of  $\text{rad/s}^2$  for angular acceleration, we need to convert from rpm to rad/s:

$$\Delta\omega = 250 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ s}} = 26.2 \frac{\text{rad}}{\text{s}}.$$

Entering this quantity into the expression for  $\alpha$ , we get

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{26.2 \text{ rad/s}}{5.00 \text{ s}} = 5.24 \text{ rad/s}^2.$$

- b. Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that  $\Delta\omega$  is  $-26.2 \text{ rad/s}$ , and  $\alpha$  is given to be  $-87.3 \text{ rad/s}^2$ . Thus,

$$\Delta t = \frac{-26.2 \text{ rad/s}}{-87.3 \text{ rad/s}^2} = 0.300 \text{ s}.$$

### Significance

Note that the angular acceleration as the mechanic spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero.



**10.1 Check Your Understanding** The fan blades on a turbofan jet engine (shown below) accelerate from rest up to a rotation rate of 40.0 rev/s in 20 s. The increase in angular velocity of the fan is constant in time. (The GE90-110B1 turbofan engine mounted on a Boeing 777, as shown, is currently the largest turbofan engine in the world, capable of thrusts of 330–510 kN.)

- What is the average angular acceleration?
- What is the instantaneous angular acceleration at any time during the first 20 s?



**Figure 10.9** (credit: “Bubinator”/ Wikimedia Commons)

### Example 10.3

#### Wind Turbine

A wind turbine (**Figure 10.10**) in a wind farm is being shut down for maintenance. It takes 30 s for the turbine to go from its operating angular velocity to a complete stop in which the angular velocity function is  $\omega(t) = [(ts^{-1} - 30.0)^2 / 100.0] \text{ rad/s}$ . If the turbine is rotating counterclockwise looking into the page, (a) what are the directions of the angular velocity and acceleration vectors? (b) What is the average angular acceleration? (c) What is the instantaneous angular acceleration at  $t = 0.0, 15.0, 30.0$  s?



**Figure 10.10** A wind turbine that is rotating counterclockwise, as seen head on.

### Strategy

- We are given the rotational sense of the turbine, which is counterclockwise in the plane of the page. Using the right hand rule (Figure 10.5), we can establish the directions of the angular velocity and acceleration vectors.
- We calculate the initial and final angular velocities to get the average angular acceleration. We establish the sign of the angular acceleration from the results in (a).
- We are given the functional form of the angular velocity, so we can find the functional form of the angular acceleration function by taking its derivative with respect to time.

### Solution

- Since the turbine is rotating counterclockwise, angular velocity  $\vec{\omega}$  points out of the page. But since the angular velocity is decreasing, the angular acceleration  $\vec{\alpha}$  points into the page, in the opposite sense to the angular velocity.
- The initial angular velocity of the turbine, setting  $t = 0$ , is  $\omega = 9.0 \text{ rad/s}$ . The final angular velocity is zero, so the average angular acceleration is

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t - t_0} = \frac{0 - 9.0 \text{ rad/s}}{30.0 - 0 \text{ s}} = -0.3 \text{ rad/s}^2.$$

- Taking the derivative of the angular velocity with respect to time gives  $\alpha = \frac{d\omega}{dt} = (t - 30.0)/50.0 \text{ rad/s}^2$

$$\alpha(0.0 \text{ s}) = -0.6 \text{ rad/s}^2, \alpha(15.0 \text{ s}) = -0.3 \text{ rad/s}^2, \text{ and } \alpha(30.0 \text{ s}) = 0 \text{ rad/s}^2.$$

### Significance

We found from the calculations in (a) and (b) that the angular acceleration  $\alpha$  and the average angular acceleration  $\bar{\alpha}$  are negative. The turbine has an angular acceleration in the opposite sense to its angular velocity.

We now have a basic vocabulary for discussing fixed-axis rotational kinematics and relationships between rotational variables. We discuss more definitions and connections in the next section.

## 10.2 | Rotation with Constant Angular Acceleration

### Learning Objectives

By the end of this section, you will be able to:

- Derive the kinematic equations for rotational motion with constant angular acceleration
- Select from the kinematic equations for rotational motion with constant angular acceleration the appropriate equations to solve for unknowns in the analysis of systems undergoing fixed-axis rotation
- Use solutions found with the kinematic equations to verify the graphical analysis of fixed-axis rotation with constant angular acceleration

In the preceding section, we defined the rotational variables of angular displacement, angular velocity, and angular acceleration. In this section, we work with these definitions to derive relationships among these variables and use these relationships to analyze rotational motion for a rigid body about a fixed axis under a constant angular acceleration. This analysis forms the basis for rotational kinematics. If the angular acceleration is constant, the equations of rotational kinematics simplify, similar to the equations of linear kinematics discussed in **Motion along a Straight Line** and **Motion in Two and Three Dimensions**. We can then use this simplified set of equations to describe many applications in physics and engineering where the angular acceleration of the system is constant. Rotational kinematics is also a prerequisite to the discussion of rotational dynamics later in this chapter.

### Kinematics of Rotational Motion

Using our intuition, we can begin to see how the rotational quantities  $\theta$ ,  $\omega$ ,  $\alpha$ , and  $t$  are related to one another. For example, we saw in the preceding section that if a flywheel has an angular acceleration in the same direction as its angular velocity vector, its angular velocity increases with time and its angular displacement also increases. On the contrary, if the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. We can describe these physical situations and many others with a consistent set of rotational kinematic equations under a constant angular acceleration. The method to investigate rotational motion in this way is called **kinematics of rotational motion**.

To begin, we note that if the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half the sum of the initial and final values:

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2}. \quad (10.9)$$

From the definition of the average angular velocity, we can find an equation that relates the angular position, average angular velocity, and time:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}.$$

Solving for  $\theta$ , we have

$$\theta_f = \theta_0 + \bar{\omega} t, \quad (10.10)$$

where we have set  $t_0 = 0$ . This equation can be very useful if we know the average angular velocity of the system. Then we could find the angular displacement over a given time period. Next, we find an equation relating  $\omega$ ,  $\alpha$ , and  $t$ . To determine this equation, we start with the definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt}.$$

We rearrange this to get  $\alpha dt = d\omega$  and then we integrate both sides of this equation from initial values to final values, that is, from  $t_0$  to  $t$  and  $\omega_0$  to  $\omega_f$ . In uniform rotational motion, the angular acceleration is constant so it can be pulled out of the integral, yielding two definite integrals:

$$\alpha \int_{t_0}^t dt' = \int_{\omega_0}^{\omega_f} d\omega.$$

Setting  $t_0 = 0$ , we have

$$\alpha t = \omega_f - \omega_0.$$

We rearrange this to obtain

$$\omega_f = \omega_0 + \alpha t, \quad (10.11)$$

where  $\omega_0$  is the initial angular velocity. **Equation 10.11** is the rotational counterpart to the linear kinematics equation  $v_f = v_0 + at$ . With **Equation 10.11**, we can find the angular velocity of an object at any specified time  $t$  given the initial angular velocity and the angular acceleration.

Let's now do a similar treatment starting with the equation  $\omega = \frac{d\theta}{dt}$ . We rearrange it to obtain  $\omega dt = d\theta$  and integrate both sides from initial to final values again, noting that the angular acceleration is constant and does not have a time dependence. However, this time, the angular velocity is not constant (in general), so we substitute in what we derived above:

$$\int_{t_0}^{t_f} (\omega_0 + \alpha t') dt' = \int_{\theta_0}^{\theta_f} d\theta;$$

$$\int_{t_0}^t \omega_0 dt + \int_{t_0}^t \alpha t dt = \int_{\theta_0}^{\theta_f} d\theta = \left[ \omega_0 t' + \alpha \left( \frac{(t')^2}{2} \right) \right]_{t_0}^t = \omega_0 t + \alpha \left( \frac{t^2}{2} \right) = \theta_f - \theta_0,$$

where we have set  $t_0 = 0$ . Now we rearrange to obtain

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2. \quad (10.12)$$

**Equation 10.12** is the rotational counterpart to the linear kinematics equation found in **Motion Along a Straight Line** for position as a function of time. This equation gives us the angular position of a rotating rigid body at any time  $t$  given the initial conditions (initial angular position and initial angular velocity) and the angular acceleration.

We can find an equation that is independent of time by solving for  $t$  in **Equation 10.11** and substituting into **Equation 10.12**. **Equation 10.12** becomes

$$\begin{aligned}
 \theta_f &= \theta_0 + \omega_0 \left( \frac{\omega_f - \omega_0}{\alpha} \right) + \frac{1}{2} \alpha \left( \frac{\omega_f - \omega_0}{\alpha} \right)^2 \\
 &= \theta_0 + \frac{\omega_0 \omega_f}{\alpha} - \frac{\omega_0^2}{\alpha} + \frac{1}{2} \frac{\omega_f^2}{\alpha} - \frac{\omega_0 \omega_f}{\alpha} + \frac{1}{2} \frac{\omega_0^2}{\alpha} \\
 &= \theta_0 + \frac{1}{2} \frac{\omega_f^2}{\alpha} - \frac{1}{2} \frac{\omega_0^2}{\alpha}, \\
 \theta_f - \theta_0 &= \frac{\omega_f^2 - \omega_0^2}{2\alpha}
 \end{aligned}$$

or

$$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta). \quad (10.13)$$

**Equation 10.10** through **Equation 10.13** describe fixed-axis rotation for constant acceleration and are summarized in **Table 10.1**.

Angular displacement from average angular velocity	$\theta_f = \theta_0 + \bar{\omega} t$
Angular velocity from angular acceleration	$\omega_f = \omega_0 + \alpha t$
Angular displacement from angular velocity and angular acceleration	$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$
Angular velocity from angular displacement and angular acceleration	$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$

**Table 10.1 Kinematic Equations**

## Applying the Equations for Rotational Motion

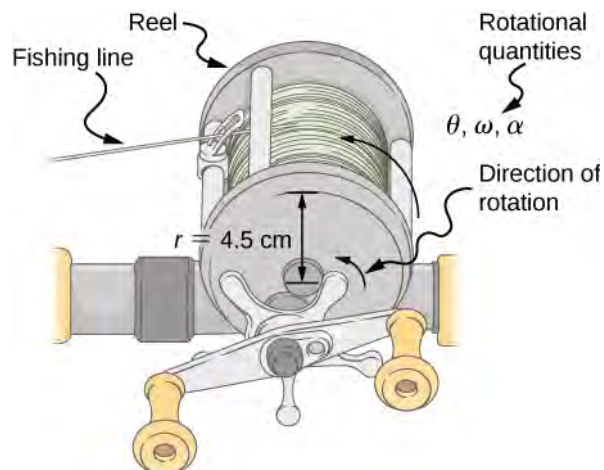
Now we can apply the key kinematic relations for rotational motion to some simple examples to get a feel for how the equations can be applied to everyday situations.

### Example 10.4

#### Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat, pulling the fishing line from his fishing reel. The whole system is initially at rest, and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of  $110 \text{ rad/s}^2$  for 2.00 s (**Figure 10.11**).

- What is the final angular velocity of the reel after 2 s?
- How many revolutions does the reel make?



**Figure 10.11** Fishing line coming off a rotating reel moves linearly.

### Strategy

Identify the knowns and compare with the kinematic equations for constant acceleration. Look for the appropriate equation that can be solved for the unknown, using the knowns given in the problem description.

### Solution

- a. We are given  $\alpha$  and  $t$  and want to determine  $\omega$ . The most straightforward equation to use is  $\omega_f = \omega_0 + \alpha t$ , since all terms are known besides the unknown variable we are looking for. We are given that  $\omega_0 = 0$  (it starts from rest), so

$$\omega_f = 0 + (110 \text{ rad/s}^2)(2.00 \text{ s}) = 220 \text{ rad/s.}$$

- b. We are asked to find the number of revolutions. Because  $1 \text{ rev} = 2\pi \text{ rad}$ , we can find the number of revolutions by finding  $\theta$  in radians. We are given  $\alpha$  and  $t$ , and we know  $\omega_0$  is zero, so we can obtain  $\theta$  by using

$$\begin{aligned} \theta_f &= \theta_i + \omega_i t + \frac{1}{2} \alpha t^2 \\ &= 0 + 0 + (0.500)(110 \text{ rad/s}^2)(2.00 \text{ s})^2 = 220 \text{ rad.} \end{aligned}$$

Converting radians to revolutions gives

$$\text{Number of rev} = (220 \text{ rad}) \frac{1 \text{ rev}}{2\pi \text{ rad}} = 35.0 \text{ rev.}$$

### Significance

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.)

In the preceding example, we considered a fishing reel with a positive angular acceleration. Now let us consider what happens with a negative angular acceleration.

## Example 10.5

### Calculating the Duration When the Fishing Reel Slows Down and Stops

Now the fisherman applies a brake to the spinning reel, achieving an angular acceleration of  $-300 \text{ rad/s}^2$ . How

long does it take the reel to come to a stop?

### Strategy

We are asked to find the time  $t$  for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is  $\omega_0 = 220 \text{ rad/s}$  and the final angular velocity  $\omega$  is zero. The angular acceleration is given as  $\alpha = -300 \text{ rad/s}^2$ . Examining the available equations, we see all quantities but  $t$  are known in  $\omega_f = \omega_0 + \alpha t$ , making it easiest to use this equation.

### Solution

The equation states

$$\omega_f = \omega_0 + \alpha t.$$

We solve the equation algebraically for  $t$  and then substitute the known values as usual, yielding

$$t = \frac{\omega_f - \omega_0}{\alpha} = \frac{0 - 220.0 \text{ rad/s}}{-300.0 \text{ rad/s}^2} = 0.733 \text{ s}.$$

### Significance

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish is slower, requiring a smaller acceleration.



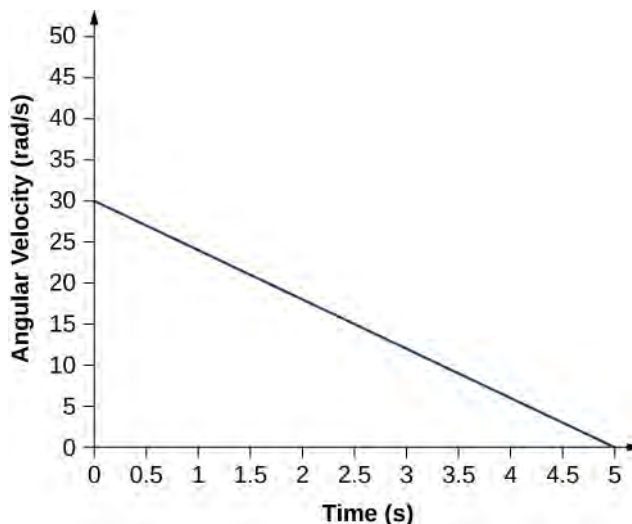
**10.2 Check Your Understanding** A centrifuge used in DNA extraction spins at a maximum rate of 7000 rpm, producing a “g-force” on the sample that is 6000 times the force of gravity. If the centrifuge takes 10 seconds to come to rest from the maximum spin rate: (a) What is the angular acceleration of the centrifuge? (b) What is the angular displacement of the centrifuge during this time?

## Example 10.6

### Angular Acceleration of a Propeller

**Figure 10.12** shows a graph of the angular velocity of a propeller on an aircraft as a function of time. Its angular velocity starts at  $30 \text{ rad/s}$  and drops linearly to  $0 \text{ rad/s}$  over the course of 5 seconds. (a) Find the angular acceleration of the object and verify the result using the kinematic equations. (b) Find the angle through which the propeller rotates during these 5 seconds and verify your result using the kinematic equations.





**Figure 10.12** A graph of the angular velocity of a propeller versus time.

### Strategy

- Since the angular velocity varies linearly with time, we know that the angular acceleration is constant and does not depend on the time variable. The angular acceleration is the slope of the angular velocity vs. time graph,  $\alpha = \frac{d\omega}{dt}$ . To calculate the slope, we read directly from **Figure 10.12**, and see that  $\omega_0 = 30 \text{ rad/s}$  at  $t = 0 \text{ s}$  and  $\omega_f = 0 \text{ rad/s}$  at  $t = 5 \text{ s}$ . Then, we can verify the result using  $\omega = \omega_0 + \alpha t$ .
- We use the equation  $\omega = \frac{d\theta}{dt}$ ; since the time derivative of the angle is the angular velocity, we can find the angular displacement by integrating the angular velocity, which from the figure means taking the area under the angular velocity graph. In other words:

$$\int_{\theta_0}^{\theta_f} d\theta = \theta_f - \theta_0 = \int_{t_0}^{t_f} \omega(t) dt.$$

Then we use the kinematic equations for constant acceleration to verify the result.

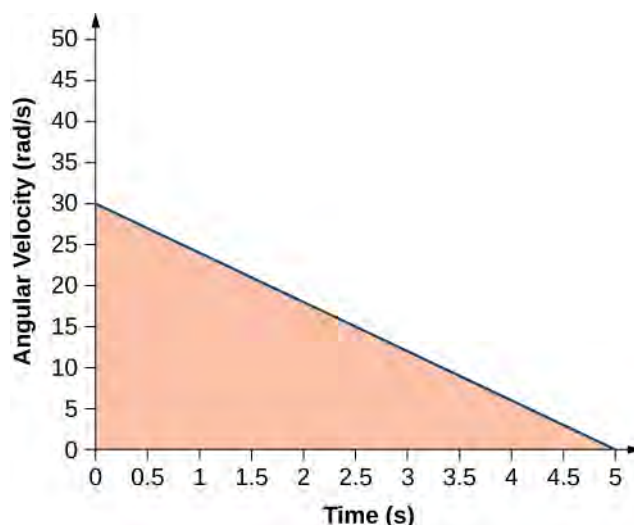
### Solution

- Calculating the slope, we get

$$\alpha = \frac{\omega - \omega_0}{t - t_0} = \frac{(0 - 30.0) \text{ rad/s}}{(5.0 - 0) \text{ s}} = -6.0 \text{ rad/s}^2.$$

We see that this is exactly **Equation 10.11** with a little rearranging of terms.

- We can find the area under the curve by calculating the area of the right triangle, as shown in **Figure 10.13**.



**Figure 10.13** The area under the curve is the area of the right triangle.

$$\Delta\theta = \text{area}(\text{triangle});$$

$$\Delta\theta = \frac{1}{2}(30 \text{ rad/s})(5 \text{ s}) = 75 \text{ rad}.$$

We verify the solution using **Equation 10.12**:

$$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}at^2.$$

Setting  $\theta_0 = 0$ , we have

$$\theta_f = (30.0 \text{ rad/s})(5.0 \text{ s}) + \frac{1}{2}(-6.0 \text{ rad/s}^2)(5.0 \text{ rad/s})^2 = 150.0 - 75.0 = 75.0 \text{ rad}.$$

This verifies the solution found from finding the area under the curve.

### Significance

We see from part (b) that there are alternative approaches to analyzing fixed-axis rotation with constant acceleration. We started with a graphical approach and verified the solution using the rotational kinematic equations. Since  $\alpha = \frac{d\omega}{dt}$ , we could do the same graphical analysis on an angular acceleration-vs.-time curve.

The area under an  $\alpha$ -vs.- $t$  curve gives us the change in angular velocity. Since the angular acceleration is constant in this section, this is a straightforward exercise.

## 10.3 | Relating Angular and Translational Quantities

### Learning Objectives

By the end of this section, you will be able to:

- Given the linear kinematic equation, write the corresponding rotational kinematic equation
- Calculate the linear distances, velocities, and accelerations of points on a rotating system given the angular velocities and accelerations

In this section, we relate each of the rotational variables to the translational variables defined in **Motion Along a Straight Line** and **Motion in Two and Three Dimensions**. This will complete our ability to describe rigid-body rotations.

## Angular vs. Linear Variables

In **Rotational Variables**, we introduced angular variables. If we compare the rotational definitions with the definitions of linear kinematic variables from **Motion Along a Straight Line** and **Motion in Two and Three Dimensions**, we find that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as we can see when we write them side by side:

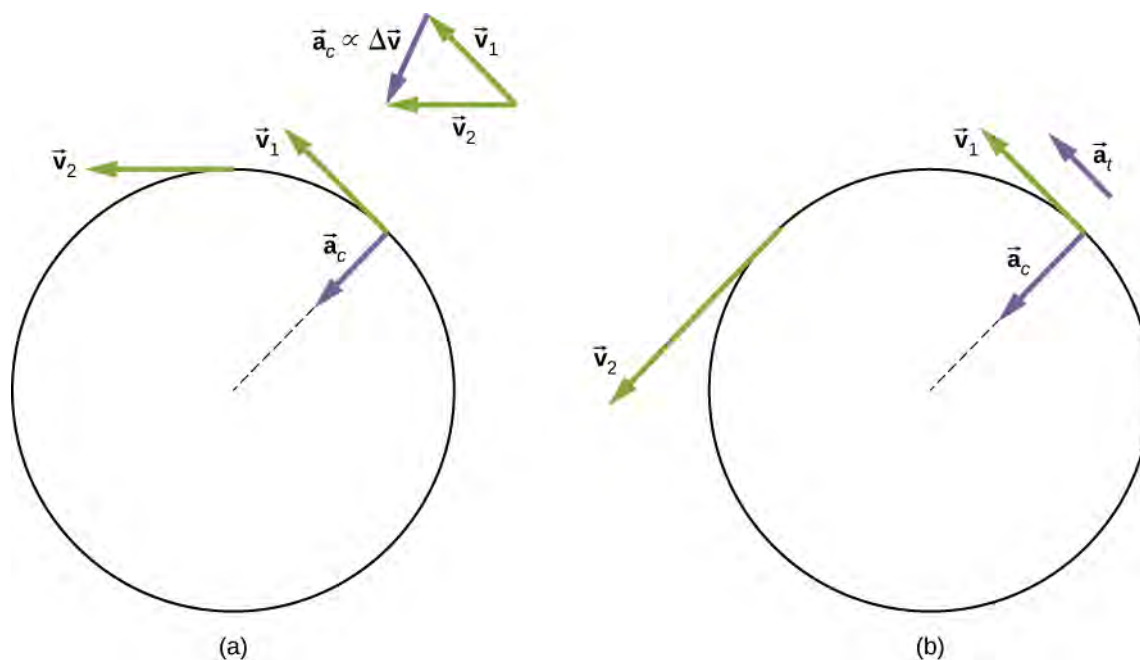
	Linear	Rotational
Position	$x$	$\theta$
Velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

Let's compare the linear and rotational variables individually. The linear variable of position has physical units of meters, whereas the angular position variable has dimensionless units of radians, as can be seen from the definition of  $\theta = \frac{s}{r}$ , which is the ratio of two lengths. The linear velocity has units of m/s, and its counterpart, the angular velocity, has units of rad/s. In **Rotational Variables**, we saw in the case of circular motion that the linear tangential speed of a particle at a radius  $r$  from the axis of rotation is related to the angular velocity by the relation  $v_t = r\omega$ . This could also apply to points on a rigid body rotating about a fixed axis. Here, we consider only circular motion. In circular motion, both uniform and nonuniform, there exists a centripetal acceleration (**Motion in Two and Three Dimensions**). The centripetal acceleration vector points inward from the particle executing circular motion toward the axis of rotation. The derivation of the magnitude of the centripetal acceleration is given in **Motion in Two and Three Dimensions**. From that derivation, the magnitude of the centripetal acceleration was found to be

$$a_c = \frac{v_t^2}{r}, \quad (10.14)$$

where  $r$  is the radius of the circle.

Thus, in uniform circular motion when the angular velocity is constant and the angular acceleration is zero, we have a linear acceleration—that is, centripetal acceleration—since the tangential speed in **Equation 10.14** is a constant. If nonuniform circular motion is present, the rotating system has an angular acceleration, and we have both a linear centripetal acceleration that is changing (because  $v_t$  is changing) as well as a linear tangential acceleration. These relationships are shown in **Figure 10.14**, where we show the centripetal and tangential accelerations for uniform and nonuniform circular motion.



**Figure 10.14** (a) Uniform circular motion: The centripetal acceleration  $a_c$  has its vector inward toward the axis of rotation. There is no tangential acceleration. (b) Nonuniform circular motion: An angular acceleration produces an inward centripetal acceleration that is changing in magnitude, plus a tangential acceleration  $a_t$ .

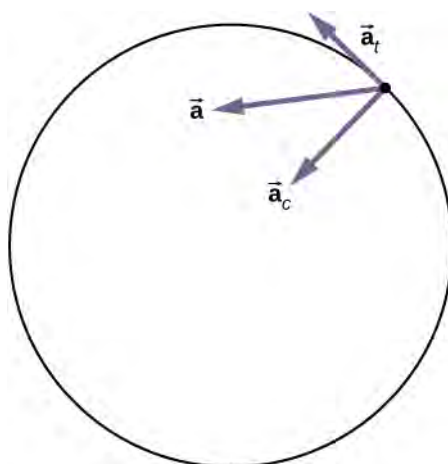
The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors  $\vec{a}_t$  and  $\vec{a}_c$  are always perpendicular to each other, as seen in **Figure 10.14**. To complete this description, we can assign a **total linear acceleration** vector to a point on a rotating rigid body or a particle executing circular motion at a radius  $r$  from a fixed axis. The total linear acceleration vector  $\vec{a}$  is the vector sum of the centripetal and tangential accelerations,

$$\vec{a} = \vec{a}_c + \vec{a}_t. \quad (10.15)$$

The total linear acceleration vector in the case of nonuniform circular motion points at an angle between the centripetal and tangential acceleration vectors, as shown in **Figure 10.15**. Since  $\vec{a}_c \perp \vec{a}_t$ , the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2}.$$

Note that if the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration.



**Figure 10.15** A particle is executing circular motion and has an angular acceleration. The total linear acceleration of the particle is the vector sum of the centripetal acceleration and tangential acceleration vectors. The total linear acceleration vector is at an angle in between the centripetal and tangential accelerations.

## Relationships between Rotational and Translational Motion

We can look at two relationships between rotational and translational motion.

1. Generally speaking, the linear kinematic equations have their rotational counterparts. **Table 10.2** lists the four linear kinematic equations and the corresponding rotational counterpart. The two sets of equations look similar to each other, but describe two different physical situations, that is, rotation and translation.

Rotational	Translational
$\theta_f = \theta_0 + \bar{\omega} t$	$x = x_0 + \bar{v} t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta x)$

**Table 10.2 Rotational and Translational Kinematic Equations**

2. The second correspondence has to do with relating linear and rotational variables in the special case of circular motion. This is shown in **Table 10.3**, where in the third column, we have listed the connecting equation that relates the linear variable to the rotational variable. The rotational variables of angular velocity and acceleration have subscripts that indicate their definition in circular motion.

Rotational	Translational	Relationship ( $r = \text{radius}$ )
$\theta$	$s$	$\theta = \frac{s}{r}$
$\omega$	$v_t$	$\omega = \frac{v_t}{r}$
$\alpha$	$a_t$	$\alpha = \frac{a_t}{r}$

**Table 10.3 Rotational and Translational Quantities: Circular Motion**

Rotational	Translational	Relationship ( $r = \text{radius}$ )
	$a_c$	$a_c = \frac{v_t^2}{r}$

**Table 10.3 Rotational and Translational Quantities: Circular Motion**

## Example 10.7

### Linear Acceleration of a Centrifuge

A centrifuge has a radius of 20 cm and accelerates from a maximum rotation rate of 10,000 rpm to rest in 30 seconds under a constant angular acceleration. It is rotating counterclockwise. What is the magnitude of the total acceleration of a point at the tip of the centrifuge at  $t = 29.0$ s? What is the direction of the total acceleration vector?

### Strategy

With the information given, we can calculate the angular acceleration, which then will allow us to find the tangential acceleration. We can find the centripetal acceleration at  $t = 0$  by calculating the tangential speed at this time. With the magnitudes of the accelerations, we can calculate the total linear acceleration. From the description of the rotation in the problem, we can sketch the direction of the total acceleration vector.

### Solution

The angular acceleration is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - (1.0 \times 10^4)2\pi/60.0 \text{ s(rad/s)}}{30.0 \text{ s}} = -34.9 \text{ rad/s}^2.$$

Therefore, the tangential acceleration is

$$a_t = r\alpha = 0.2 \text{ m}(-34.9 \text{ rad/s}^2) = -7.0 \text{ m/s}^2.$$

The angular velocity at  $t = 29.0$  s is

$$\begin{aligned}\omega &= \omega_0 + \alpha t = 1.0 \times 10^4 \left( \frac{2\pi}{60.0 \text{ s}} \right) + (-34.9 \text{ rad/s}^2)(29.0 \text{ s}) \\ &= 1047.2 \text{ rad/s} - 1012.71 = 35.1 \text{ rad/s}.\end{aligned}$$

Thus, the tangential speed at  $t = 29.0$  s is

$$v_t = r\omega = 0.2 \text{ m}(35.1 \text{ rad/s}) = 7.0 \text{ m/s}.$$

We can now calculate the centripetal acceleration at  $t = 29.0$  s:

$$a_c = \frac{v_t^2}{r} = \frac{(7.0 \text{ m/s})^2}{0.2 \text{ m}} = 245.0 \text{ m/s}^2.$$

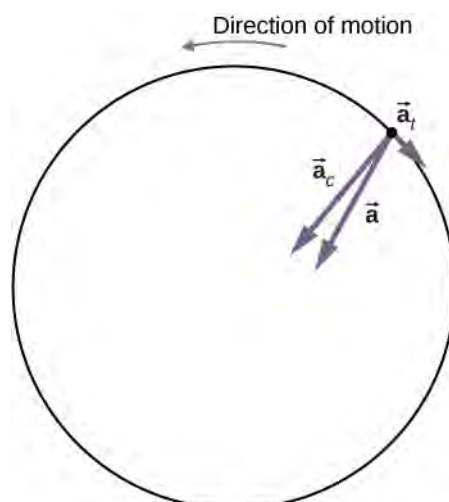
Since the two acceleration vectors are perpendicular to each other, the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2} = \sqrt{(245.0)^2 + (-7.0)^2} = 245.1 \text{ m/s}^2.$$

Since the centrifuge has a negative angular acceleration, it is slowing down. The total acceleration vector is as shown in **Figure 10.16**. The angle with respect to the centripetal acceleration vector is

$$\theta = \tan^{-1} \frac{-7.0}{245.0} = -1.6^\circ.$$

The negative sign means that the total acceleration vector is angled toward the clockwise direction.



**Figure 10.16** The centripetal, tangential, and total acceleration vectors. The centrifuge is slowing down, so the tangential acceleration is clockwise, opposite the direction of rotation (counterclockwise).

### Significance

From **Figure 10.16**, we see that the tangential acceleration vector is opposite the direction of rotation. The magnitude of the tangential acceleration is much smaller than the centripetal acceleration, so the total linear acceleration vector will make a very small angle with respect to the centripetal acceleration vector.



**10.3 Check Your Understanding** A boy jumps on a merry-go-round with a radius of 5 m that is at rest. It starts accelerating at a constant rate up to an angular velocity of 5 rad/s in 20 seconds. What is the distance travelled by the boy?



Check out this **PhET simulation (<https://openstaxcollege.org//21rotatingdisk>)** to change the parameters of a rotating disk (the initial angle, angular velocity, and angular acceleration), and place bugs at different radial distances from the axis. The simulation then lets you explore how circular motion relates to the bugs'  $xy$ -position, velocity, and acceleration using vectors or graphs.

## 10.4 | Moment of Inertia and Rotational Kinetic Energy

### Learning Objectives

By the end of this section, you will be able to:

- Describe the differences between rotational and translational kinetic energy
- Define the physical concept of moment of inertia in terms of the mass distribution from the rotational axis
- Explain how the moment of inertia of rigid bodies affects their rotational kinetic energy
- Use conservation of mechanical energy to analyze systems undergoing both rotation and translation
- Calculate the angular velocity of a rotating system when there are energy losses due to nonconservative forces

So far in this chapter, we have been working with rotational kinematics: the description of motion for a rotating rigid body

with a fixed axis of rotation. In this section, we define two new quantities that are helpful for analyzing properties of rotating objects: moment of inertia and rotational kinetic energy. With these properties defined, we will have two important tools we need for analyzing rotational dynamics.

## Rotational Kinetic Energy

Any moving object has kinetic energy. We know how to calculate this for a body undergoing translational motion, but how about for a rigid body undergoing rotation? This might seem complicated because each point on the rigid body has a different velocity. However, we can make use of angular velocity—which is the same for the entire rigid body—to express the kinetic energy for a rotating object. **Figure 10.17** shows an example of a very energetic rotating body: an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are generated as the grindstone does its work. This system has considerable energy, some of it in the form of heat, light, sound, and vibration. However, most of this energy is in the form of **rotational kinetic energy**.



**Figure 10.17** The rotational kinetic energy of the grindstone is converted to heat, light, sound, and vibration. (credit: Zachary David Bell, US Navy)

Energy in rotational motion is not a new form of energy; rather, it is the energy associated with rotational motion, the same as kinetic energy in translational motion. However, because kinetic energy is given by  $K = \frac{1}{2}mv^2$ , and velocity is a quantity that is different for every point on a rotating body about an axis, it makes sense to find a way to write kinetic energy in terms of the variable  $\omega$ , which is the same for all points on a rigid rotating body. For a single particle rotating around a fixed axis, this is straightforward to calculate. We can relate the angular velocity to the magnitude of the translational velocity using the relation  $v_t = \omega r$ , where  $r$  is the distance of the particle from the axis of rotation and  $v_t$  is its tangential speed. Substituting into the equation for kinetic energy, we find

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2.$$

In the case of a rigid rotating body, we can divide up any body into a large number of smaller masses, each with a mass  $m_j$  and distance to the axis of rotation  $r_j$ , such that the total mass of the body is equal to the sum of the individual masses:

$M = \sum_j m_j$ . Each smaller mass has tangential speed  $v_j$ , where we have dropped the subscript  $t$  for the moment. The total

kinetic energy of the rigid rotating body is

$$K = \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j (r_j \omega_j)^2$$



and since  $\omega_j = \omega$  for all masses,

$$K = \frac{1}{2} \left( \sum_j m_j r_j^2 \right) \omega^2. \quad (10.16)$$

The units of **Equation 10.16** are joules (J). The equation in this form is complete, but awkward; we need to find a way to generalize it.

## Moment of Inertia

If we compare **Equation 10.16** to the way we wrote kinetic energy in **Work and Kinetic Energy**,  $\left(\frac{1}{2}mv^2\right)$ , this suggests we have a new rotational variable to add to our list of our relations between rotational and translational variables. The quantity  $\sum_j m_j r_j^2$  is the counterpart for mass in the equation for rotational kinetic energy. This is an important new term for rotational motion. This quantity is called the **moment of inertia**  $I$ , with units of  $\text{kg} \cdot \text{m}^2$ :

$$I = \sum_j m_j r_j^2. \quad (10.17)$$

For now, we leave the expression in summation form, representing the moment of inertia of a system of point particles rotating about a fixed axis. We note that the moment of inertia of a single point particle about a fixed axis is simply  $mr^2$ , with  $r$  being the distance from the point particle to the axis of rotation. In the next section, we explore the integral form of this equation, which can be used to calculate the moment of inertia of some regular-shaped rigid bodies.

The moment of inertia is the quantitative measure of rotational inertia, just as in translational motion, and mass is the quantitative measure of linear inertia—that is, the more massive an object is, the more inertia it has, and the greater is its resistance to change in linear velocity. Similarly, the greater the moment of inertia of a rigid body or system of particles, the greater is its resistance to change in angular velocity about a fixed axis of rotation. It is interesting to see how the moment of inertia varies with  $r$ , the distance to the axis of rotation of the mass particles in **Equation 10.17**. Rigid bodies and systems of particles with more mass concentrated at a greater distance from the axis of rotation have greater moments of inertia than bodies and systems of the same mass, but concentrated near the axis of rotation. In this way, we can see that a hollow cylinder has more rotational inertia than a solid cylinder of the same mass when rotating about an axis through the center. Substituting **Equation 10.17** into **Equation 10.16**, the expression for the kinetic energy of a rotating rigid body becomes

$$K = \frac{1}{2} I \omega^2. \quad (10.18)$$

We see from this equation that the kinetic energy of a rotating rigid body is directly proportional to the moment of inertia and the square of the angular velocity. This is exploited in flywheel energy-storage devices, which are designed to store large amounts of rotational kinetic energy. Many carmakers are now testing flywheel energy storage devices in their automobiles, such as the flywheel, or kinetic energy recovery system, shown in **Figure 10.18**.



**Figure 10.18** A KERS (kinetic energy recovery system) flywheel used in cars. (credit: “cmonville”/Flickr)

The rotational and translational quantities for kinetic energy and inertia are summarized in **Table 10.4**. The relationship column is not included because a constant doesn't exist by which we could multiply the rotational quantity to get the translational quantity, as can be done for the variables in **Table 10.3**.

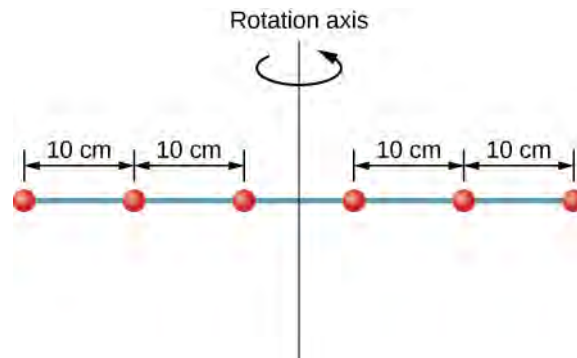
Rotational	Translational
$I = \sum_j m_j r_j^2$	$m$
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$

**Table 10.4 Rotational and Translational Kinetic Energies and Inertia**

## Example 10.8

### Moment of Inertia of a System of Particles

Six small washers are spaced 10 cm apart on a rod of negligible mass and 0.5 m in length. The mass of each washer is 20 g. The rod rotates about an axis located at 25 cm, as shown in **Figure 10.19**. (a) What is the moment of inertia of the system? (b) If the two washers closest to the axis are removed, what is the moment of inertia of the remaining four washers? (c) If the system with six washers rotates at 5 rev/s, what is its rotational kinetic energy?



**Figure 10.19** Six washers are spaced 10 cm apart on a rod of negligible mass and rotating about a vertical axis.

### Strategy

- We use the definition for moment of inertia for a system of particles and perform the summation to evaluate this quantity. The masses are all the same so we can pull that quantity in front of the summation symbol.
- We do a similar calculation.
- We insert the result from (a) into the expression for rotational kinetic energy.

### Solution

$$a. \quad I = \sum_j m_j r_j^2 = (0.02 \text{ kg})(2 \times (0.25 \text{ m})^2 + 2 \times (0.15 \text{ m})^2 + 2 \times (0.05 \text{ m})^2) = 0.0035 \text{ kg} \cdot \text{m}^2.$$

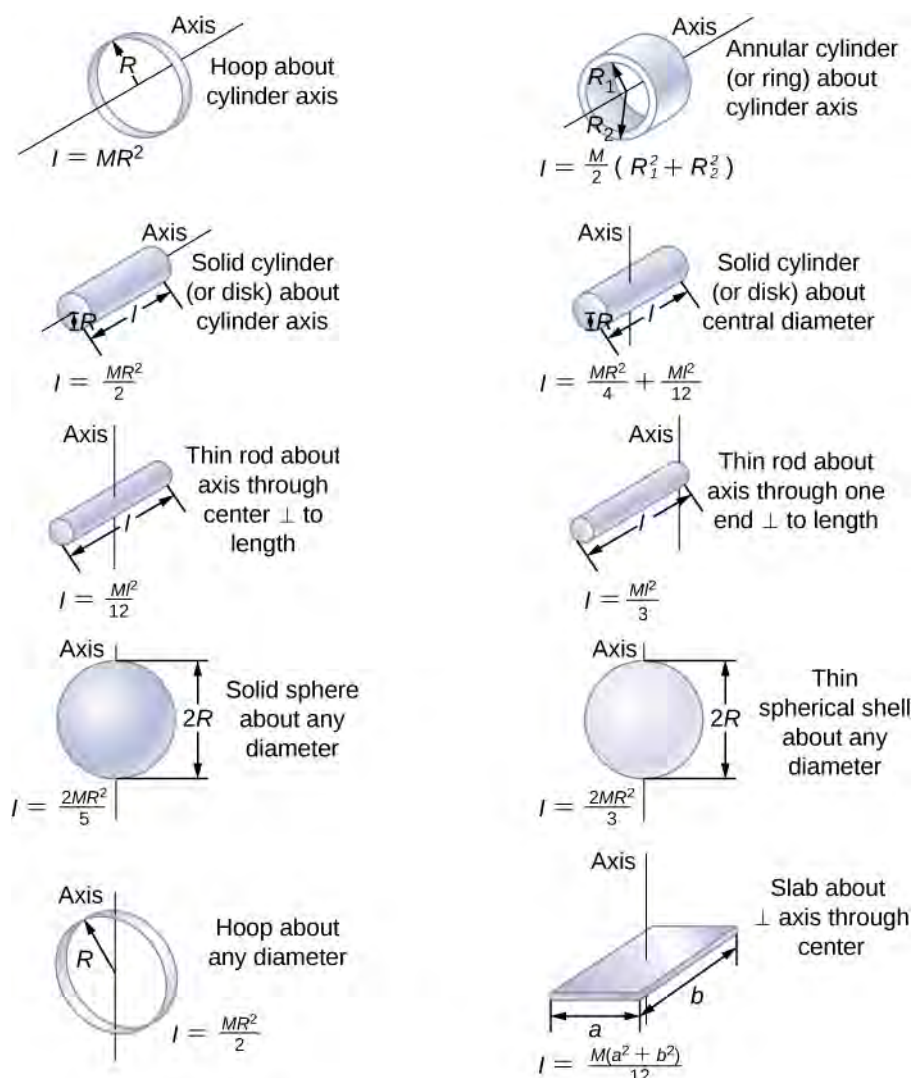
$$b. \quad I = \sum_j m_j r_j^2 = (0.02 \text{ kg})(2 \times (0.25 \text{ m})^2 + 2 \times (0.15 \text{ m})^2) = 0.0034 \text{ kg} \cdot \text{m}^2.$$

$$c. \quad K = \frac{1}{2} I \omega^2 = \frac{1}{2} (0.0035 \text{ kg} \cdot \text{m}^2) (5.0 \times 2\pi \text{ rad/s})^2 = 1.73 \text{ J}.$$

### Significance

We can see the individual contributions to the moment of inertia. The masses close to the axis of rotation have a very small contribution. When we removed them, it had a very small effect on the moment of inertia.

In the next section, we generalize the summation equation for point particles and develop a method to calculate moments of inertia for rigid bodies. For now, though, **Figure 10.20** gives values of rotational inertia for common object shapes around specified axes.



**Figure 10.20** Values of rotational inertia for common shapes of objects.

## Applying Rotational Kinetic Energy

Now let's apply the ideas of rotational kinetic energy and the moment of inertia table to get a feeling for the energy associated with a few rotating objects. The following examples will also help get you comfortable using these equations. First, let's look at a general problem-solving strategy for rotational energy.

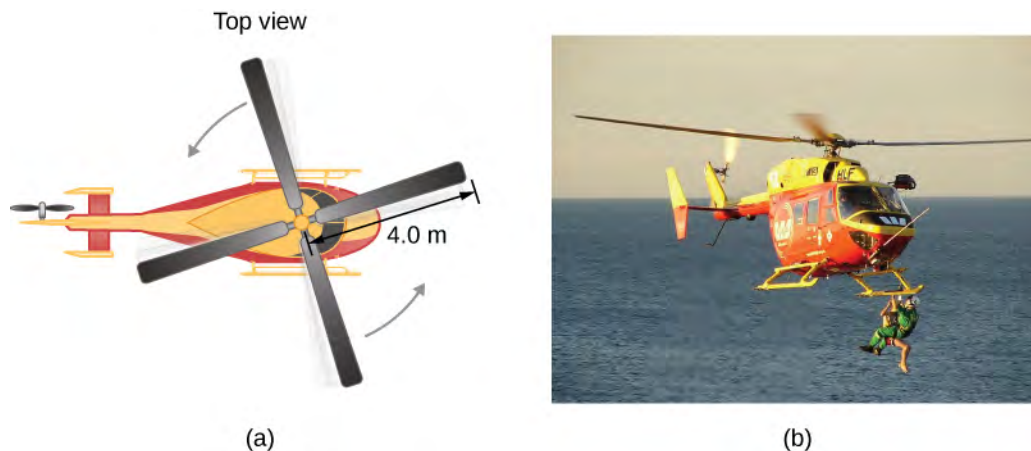
### Problem-Solving Strategy: Rotational Energy

1. Determine that energy or work is involved in the rotation.
2. Determine the system of interest. A sketch usually helps.
3. Analyze the situation to determine the types of work and energy involved.
4. If there are no losses of energy due to friction and other nonconservative forces, mechanical energy is conserved, that is,  $K_i + U_i = K_f + U_f$ .
5. If nonconservative forces are present, mechanical energy is not conserved, and other forms of energy, such as heat and light, may enter or leave the system. Determine what they are and calculate them as necessary.
6. Eliminate terms wherever possible to simplify the algebra.
7. Evaluate the numerical solution to see if it makes sense in the physical situation presented in the wording of the problem.

## Example 10.9

### Calculating Helicopter Energies

A typical small rescue helicopter has four blades: Each is 4.00 m long and has a mass of 50.0 kg (**Figure 10.21**). The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades.



**Figure 10.21** (a) Sketch of a four-blade helicopter. (b) A water rescue operation featuring a helicopter from the Auckland Westpac Rescue Helicopter Service. (credit b: “111 Emergency”/Flickr)

### Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The wording of the problem gives all the necessary constants to evaluate the expressions for the rotational and translational kinetic energies.

### Solution

- a. The rotational kinetic energy is

$$K = \frac{1}{2}I\omega^2.$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find  $K$ . The angular velocity  $\omega$  is

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \frac{2\pi \text{ rad}}{1 \text{ rev}} \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade is that of a thin rod rotated about its end, listed in **Figure 10.20**. The total  $I$  is four times this moment of inertia because there are four blades. Thus,

$$I = 4\frac{Ml^2}{3} = 4 \times \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} = 1067.0 \text{ kg} \cdot \text{m}^2.$$

Entering  $\omega$  and  $I$  into the expression for rotational kinetic energy gives

$$K = 0.5(1067 \text{ kg} \cdot \text{m}^2)(31.4 \text{ rad/s})^2 = 5.26 \times 10^5 \text{ J}.$$

- b. Entering the given values into the equation for translational kinetic energy, we obtain

$$K = \frac{1}{2}mv^2 = (0.5)(1000.0 \text{ kg})(20.0 \text{ m/s})^2 = 2.00 \times 10^5 \text{ J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy.

This ratio is

$$\frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380.$$

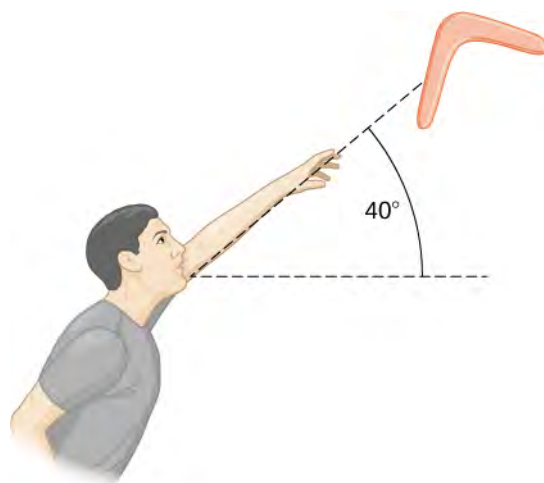
### Significance

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades.

## Example 10.10

### Energy in a Boomerang

A person hurls a boomerang into the air with a velocity of 30.0 m/s at an angle of  $40.0^\circ$  with respect to the horizontal (**Figure 10.22**). It has a mass of 1.0 kg and is rotating at 10.0 rev/s. The moment of inertia of the boomerang is given as  $I = \frac{1}{12}mL^2$  where  $L = 0.7 \text{ m}$ . (a) What is the total energy of the boomerang when it leaves the hand? (b) How high does the boomerang go from the elevation of the hand, neglecting air resistance?



**Figure 10.22** A boomerang is hurled into the air at an initial angle of  $40^\circ$ .

### Strategy

We use the definitions of rotational and linear kinetic energy to find the total energy of the system. The problem states to neglect air resistance, so we don't have to worry about energy loss. In part (b), we use conservation of mechanical energy to find the maximum height of the boomerang.

### Solution

a. Moment of inertia:  $I = \frac{1}{12}mL^2 = \frac{1}{12}(1.0 \text{ kg})(0.7 \text{ m})^2 = 0.041 \text{ kg} \cdot \text{m}^2$ .

Angular velocity:  $\omega = (10.0 \text{ rev/s})(2\pi) = 62.83 \text{ rad/s}$ .

The rotational kinetic energy is therefore

$$K_R = \frac{1}{2}(0.041 \text{ kg} \cdot \text{m}^2)(62.83 \text{ rad/s})^2 = 80.93 \text{ J}.$$

The translational kinetic energy is

$$K_T = \frac{1}{2}mv^2 = \frac{1}{2}(1.0 \text{ kg})(30.0 \text{ m/s})^2 = 450.0 \text{ J}.$$

Thus, the total energy in the boomerang is

$$K_{\text{Total}} = K_{\text{R}} + K_{\text{T}} = 80.93 + 450.0 = 530.93 \text{ J.}$$

- b. We use conservation of mechanical energy. Since the boomerang is launched at an angle, we need to write the total energies of the system in terms of its linear kinetic energies using the velocity in the  $x$ - and  $y$ -directions. The total energy when the boomerang leaves the hand is

$$E_{\text{Before}} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2.$$

The total energy at maximum height is

$$E_{\text{Final}} = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 + mgh.$$

By conservation of mechanical energy,  $E_{\text{Before}} = E_{\text{Final}}$  so we have, after canceling like terms,

$$\frac{1}{2}mv_y^2 = mgh.$$

Since  $v_y = 30.0 \text{ m/s}(\sin 40^\circ) = 19.28 \text{ m/s}$ , we find

$$h = \frac{(19.28 \text{ m/s})^2}{2(9.8 \text{ m/s}^2)} = 18.97 \text{ m.}$$

### Significance

In part (b), the solution demonstrates how energy conservation is an alternative method to solve a problem that normally would be solved using kinematics. In the absence of air resistance, the rotational kinetic energy was not a factor in the solution for the maximum height.



**10.4 Check Your Understanding** A nuclear submarine propeller has a moment of inertia of  $800.0 \text{ kg} \cdot \text{m}^2$ .

If the submerged propeller has a rotation rate of  $4.0 \text{ rev/s}$  when the engine is cut, what is the rotation rate of the propeller after  $5.0 \text{ s}$  when water resistance has taken  $50,000 \text{ J}$  out of the system?

## 10.5 | Calculating Moments of Inertia

### Learning Objectives

By the end of this section, you will be able to:

- Calculate the moment of inertia for uniformly shaped, rigid bodies
- Apply the parallel axis theorem to find the moment of inertia about any axis parallel to one already known
- Calculate the moment of inertia for compound objects

In the preceding section, we defined the moment of inertia but did not show how to calculate it. In this section, we show how to calculate the moment of inertia for several standard types of objects, as well as how to use known moments of inertia to find the moment of inertia for a shifted axis or for a compound object. This section is very useful for seeing how to apply a general equation to complex objects (a skill that is critical for more advanced physics and engineering courses).

### Moment of Inertia

We defined the moment of inertia  $I$  of an object to be  $I = \sum_i m_i r_i^2$  for all the point masses that make up the object.

Because  $r$  is the distance to the axis of rotation from each piece of mass that makes up the object, the moment of inertia



for any object depends on the chosen axis. To see this, let's take a simple example of two masses at the end of a massless (negligibly small mass) rod (**Figure 10.23**) and calculate the moment of inertia about two different axes. In this case, the summation over the masses is simple because the two masses at the end of the barbell can be approximated as point masses, and the sum therefore has only two terms.

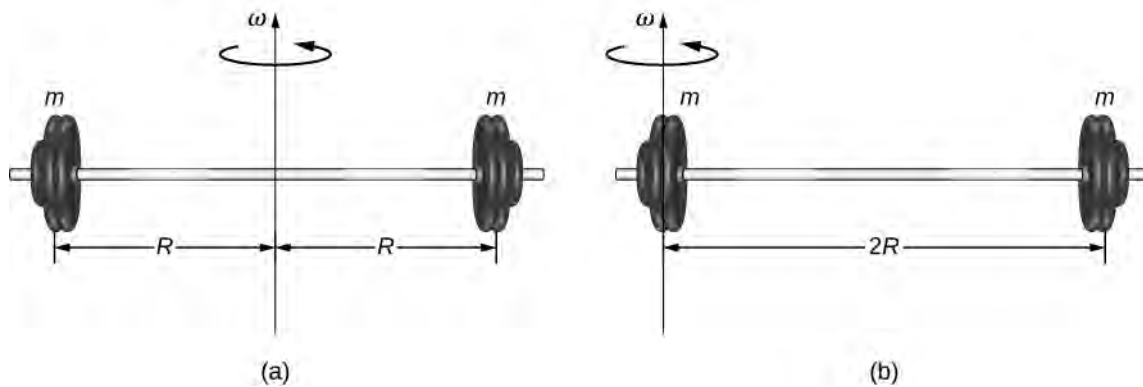
In the case with the axis in the center of the barbell, each of the two masses  $m$  is a distance  $R$  away from the axis, giving a moment of inertia of

$$I_1 = mR^2 + mR^2 = 2mR^2.$$

In the case with the axis at the end of the barbell—passing through one of the masses—the moment of inertia is

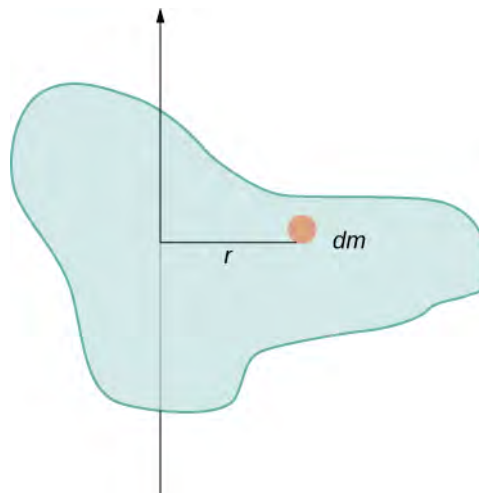
$$I_2 = m(0)^2 + m(2R)^2 = 4mR^2.$$

From this result, we can conclude that it is twice as hard to rotate the barbell about the end than about its center.



**Figure 10.23** (a) A barbell with an axis of rotation through its center; (b) a barbell with an axis of rotation through one end.

In this example, we had two point masses and the sum was simple to calculate. However, to deal with objects that are not point-like, we need to think carefully about each of the terms in the equation. The equation asks us to sum over each ‘piece of mass’ a certain distance from the axis of rotation. But what exactly does each ‘piece of mass’ mean? Recall that in our derivation of this equation, each piece of mass had the same magnitude of velocity, which means the whole piece had to have a single distance  $r$  to the axis of rotation. However, this is not possible unless we take an infinitesimally small piece of mass  $dm$ , as shown in **Figure 10.24**.



**Figure 10.24** Using an infinitesimally small piece of mass to calculate the contribution to the total moment of inertia.

The need to use an infinitesimally small piece of mass  $dm$  suggests that we can write the moment of inertia by evaluating an integral over infinitesimal masses rather than doing a discrete sum over finite masses:

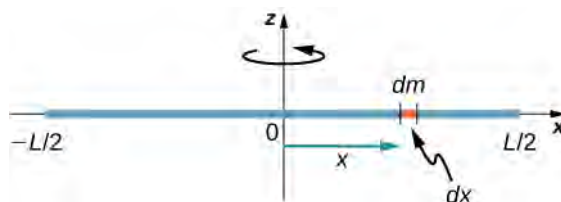


$$I = \sum_i m_i r_i^2 \text{ becomes } I = \int r^2 dm. \quad (10.19)$$

This, in fact, is the form we need to generalize the equation for complex shapes. It is best to work out specific examples in detail to get a feel for how to calculate the moment of inertia for specific shapes. This is the focus of most of the rest of this section.

### A uniform thin rod with an axis through the center

Consider a uniform (density and shape) thin rod of mass  $M$  and length  $L$  as shown in **Figure 10.25**. We want a thin rod so that we can assume the cross-sectional area of the rod is small and the rod can be thought of as a string of masses along a one-dimensional straight line. In this example, the axis of rotation is perpendicular to the rod and passes through the midpoint for simplicity. Our task is to calculate the moment of inertia about this axis. We orient the axes so that the  $z$ -axis is the axis of rotation and the  $x$ -axis passes through the length of the rod, as shown in the figure. This is a convenient choice because we can then integrate along the  $x$ -axis.



**Figure 10.25** Calculation of the moment of inertia  $I$  for a uniform thin rod about an axis through the center of the rod.

We define  $dm$  to be a small element of mass making up the rod. The moment of inertia integral is an integral over the mass distribution. However, we know how to integrate over space, not over mass. We therefore need to find a way to relate mass to spatial variables. We do this using the **linear mass density**  $\lambda$  of the object, which is the mass per unit length. Since the mass density of this object is uniform, we can write

$$\lambda = \frac{m}{l} \text{ or } m = \lambda l.$$

If we take the differential of each side of this equation, we find

$$dm = d(\lambda l) = \lambda(dl)$$

since  $\lambda$  is constant. We chose to orient the rod along the  $x$ -axis for convenience—this is where that choice becomes very helpful. Note that a piece of the rod  $dl$  lies completely along the  $x$ -axis and has a length  $dx$ ; in fact,  $dl = dx$  in this situation. We can therefore write  $dm = \lambda(dx)$ , giving us an integration variable that we know how to deal with. The distance of each piece of mass  $dm$  from the axis is given by the variable  $x$ , as shown in the figure. Putting this all together, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx.$$

The last step is to be careful about our limits of integration. The rod extends from  $x = -L/2$  to  $x = L/2$ , since the axis is in the middle of the rod at  $x = 0$ . This gives us

$$\begin{aligned} I &= \int_{-L/2}^{L/2} x^2 \lambda dx = \lambda \frac{x^3}{3} \Big|_{-L/2}^{L/2} = \lambda \left( \frac{1}{3} \right) \left[ \left( \frac{L}{2} \right)^3 - \left( \frac{-L}{2} \right)^3 \right] \\ &= \lambda \left( \frac{1}{3} \right) \frac{L^3}{8} (2) = \frac{M}{L} \left( \frac{1}{3} \right) \frac{L^3}{8} (2) = \frac{1}{12} ML^2. \end{aligned}$$

Next, we calculate the moment of inertia for the same uniform thin rod but with a different axis choice so we can compare the results. We would expect the moment of inertia to be smaller about an axis through the center of mass than the endpoint axis, just as it was for the barbell example at the start of this section. This happens because more mass is distributed farther from the axis of rotation.

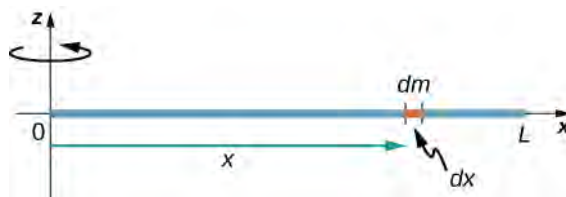
### A uniform thin rod with axis at the end

Now consider the same uniform thin rod of mass  $M$  and length  $L$ , but this time we move the axis of rotation to the end of the rod. We wish to find the moment of inertia about this new axis (Figure 10.26). The quantity  $dm$  is again defined to be a small element of mass making up the rod. Just as before, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx.$$

However, this time we have different limits of integration. The rod extends from  $x = 0$  to  $x = L$ , since the axis is at the end of the rod at  $x = 0$ . Therefore we find

$$\begin{aligned} I &= \int_0^L x^2 \lambda dx = \lambda \frac{x^3}{3} \Big|_0^L = \lambda \left( \frac{1}{3} \right) [(L)^3 - (0)^3] \\ &= \lambda \left( \frac{1}{3} \right) L^3 = \frac{M}{L} \left( \frac{1}{3} \right) L^3 = \frac{1}{3} ML^2. \end{aligned}$$



**Figure 10.26** Calculation of the moment of inertia  $I$  for a uniform thin rod about an axis through the end of the rod.

Note the rotational inertia of the rod about its endpoint is larger than the rotational inertia about its center (consistent with the barbell example) by a factor of four.

## The Parallel-Axis Theorem

The similarity between the process of finding the moment of inertia of a rod about an axis through its middle and about an axis through its end is striking, and suggests that there might be a simpler method for determining the moment of inertia for a rod about any axis parallel to the axis through the center of mass. Such an axis is called a **parallel axis**. There is a theorem for this, called the **parallel-axis theorem**, which we state here but do not derive in this text.

### Parallel-Axis Theorem

Let  $m$  be the mass of an object and let  $d$  be the distance from an axis through the object's center of mass to a new axis. Then we have

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2. \quad (10.20)$$

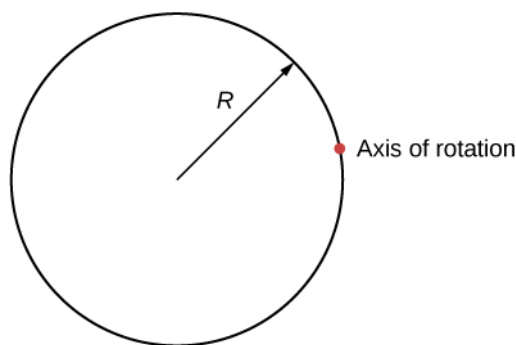
Let's apply this to the rod examples solved above:

$$I_{\text{end}} = I_{\text{center of mass}} + md^2 = \frac{1}{12} mL^2 + m \left( \frac{L}{2} \right)^2 = \left( \frac{1}{12} + \frac{1}{4} \right) mL^2 = \frac{1}{3} mL^2.$$

This result agrees with our more lengthy calculation from above. This is a useful equation that we apply in some of the examples and problems.

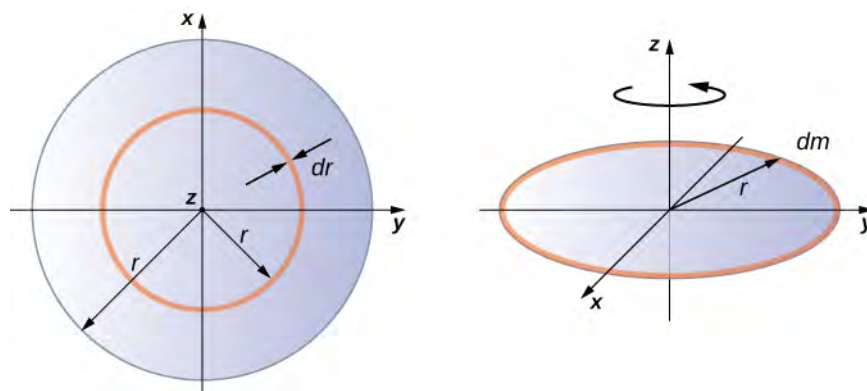


**10.5 Check Your Understanding** What is the moment of inertia of a cylinder of radius  $R$  and mass  $m$  about an axis through a point on the surface, as shown below?



### A uniform thin disk about an axis through the center

Integrating to find the moment of inertia of a two-dimensional object is a little bit trickier, but one shape is commonly done at this level of study—a uniform thin disk about an axis through its center (**Figure 10.27**).



**Figure 10.27** Calculating the moment of inertia for a thin disk about an axis through its center.

Since the disk is thin, we can take the mass as distributed entirely in the  $xy$ -plane. We again start with the relationship for the **surface mass density**, which is the mass per unit surface area. Since it is uniform, the surface mass density  $\sigma$  is constant:

$$\sigma = \frac{m}{A} \quad \text{or} \quad \sigma A = m, \quad \text{so} \quad dm = \sigma(dA).$$

Now we use a simplification for the area. The area can be thought of as made up of a series of thin rings, where each ring is a mass increment  $dm$  of radius  $r$  equidistant from the axis, as shown in part (b) of the figure. The infinitesimal area of each ring  $dA$  is therefore given by the length of each ring ( $2\pi r$ ) times the infinitesimal width of each ring  $dr$ :

$$A = \pi r^2, \quad dA = d(\pi r^2) = \pi dr^2 = 2\pi r dr.$$

The full area of the disk is then made up from adding all the thin rings with a radius range from 0 to  $R$ . This radius range then becomes our limits of integration for  $dr$ , that is, we integrate from  $r = 0$  to  $r = R$ . Putting this all together, we have

$$\begin{aligned} I &= \int_0^R r^2 \sigma(2\pi r) dr = 2\pi\sigma \int_0^R r^3 dr = 2\pi\sigma \frac{r^4}{4} \Big|_0^R = 2\pi\sigma \left( \frac{R^4}{4} - 0 \right) \\ &= 2\pi \frac{m}{A} \left( \frac{R^4}{4} \right) = 2\pi \frac{m}{\pi R^2} \left( \frac{R^4}{4} \right) = \frac{1}{2} m R^2. \end{aligned}$$

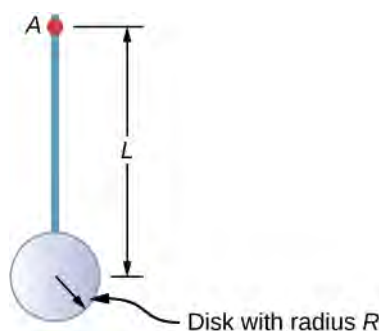
Note that this agrees with the value given in **Figure 10.20**.

### Calculating the moment of inertia for compound objects

Now consider a compound object such as that in **Figure 10.28**, which depicts a thin disk at the end of a thin rod. This cannot be easily integrated to find the moment of inertia because it is not a uniformly shaped object. However, if we go back to the initial definition of moment of inertia as a summation, we can reason that a compound object's moment of inertia can be found from the sum of each part of the object:

$$I_{\text{total}} = \sum_i I_i \quad (10.21)$$

It is important to note that the moments of inertia of the objects in **Equation 10.21** are *about a common axis*. In the case of this object, that would be a rod of length  $L$  rotating about its end, and a thin disk of radius  $R$  rotating about an axis shifted off of the center by a distance  $L + R$ , where  $R$  is the radius of the disk. Let's define the mass of the rod to be  $m_r$  and the mass of the disk to be  $m_d$ .



**Figure 10.28** Compound object consisting of a disk at the end of a rod. The axis of rotation is located at A.

The moment of inertia of the rod is simply  $\frac{1}{3}m_r L^2$ , but we have to use the parallel-axis theorem to find the moment of inertia of the disk about the axis shown. The moment of inertia of the disk about its center is  $\frac{1}{2}m_d R^2$  and we apply the parallel-axis theorem  $I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2$  to find

$$I_{\text{parallel-axis}} = \frac{1}{2}m_d R^2 + m_d (L + R)^2.$$

Adding the moment of inertia of the rod plus the moment of inertia of the disk with a shifted axis of rotation, we find the moment of inertia for the compound object to be

$$I_{\text{total}} = \frac{1}{3}m_r L^2 + \frac{1}{2}m_d R^2 + m_d (L + R)^2.$$

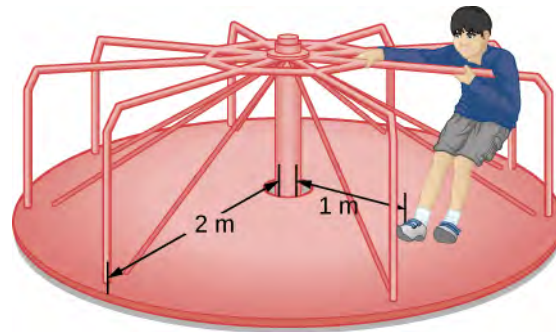
### Applying moment of inertia calculations to solve problems

Now let's examine some practical applications of moment of inertia calculations.

#### Example 10.11

##### Person on a Merry-Go-Round

A 25-kg child stands at a distance  $r = 1.0$  m from the axis of a rotating merry-go-round (**Figure 10.29**). The merry-go-round can be approximated as a uniform solid disk with a mass of 500 kg and a radius of 2.0 m. Find the moment of inertia of this system.



**Figure 10.29** Calculating the moment of inertia for a child on a merry-go-round.

### Strategy

This problem involves the calculation of a moment of inertia. We are given the mass and distance to the axis of rotation of the child as well as the mass and radius of the merry-go-round. Since the mass and size of the child are much smaller than the merry-go-round, we can approximate the child as a point mass. The notation we use is  $m_c = 25 \text{ kg}$ ,  $r_c = 1.0 \text{ m}$ ,  $m_m = 500 \text{ kg}$ ,  $r_m = 2.0 \text{ m}$ .

Our goal is to find  $I_{\text{total}} = \sum_i I_i$ .

### Solution

For the child,  $I_c = m_c r_c^2$ , and for the merry-go-round,  $I_m = \frac{1}{2} m_m r_m^2$ . Therefore

$$I_{\text{total}} = 25(1)^2 + \frac{1}{2}(500)(2)^2 = 25 + 1000 = 1025 \text{ kg} \cdot \text{m}^2.$$

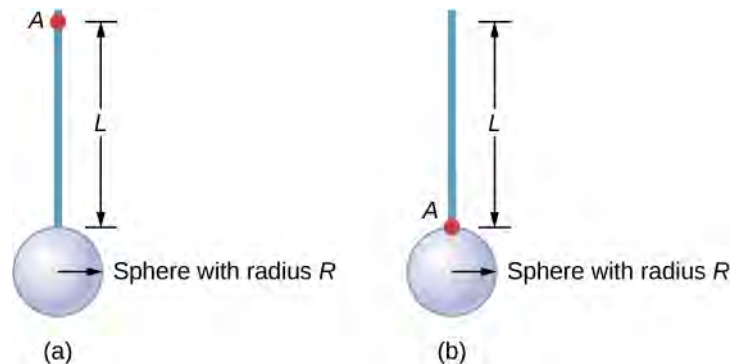
### Significance

The value should be close to the moment of inertia of the merry-go-round by itself because it has much more mass distributed away from the axis than the child does.

## Example 10.12

### Rod and Solid Sphere

Find the moment of inertia of the rod and solid sphere combination about the two axes as shown below. The rod has length 0.5 m and mass 2.0 kg. The radius of the sphere is 20.0 cm and has mass 1.0 kg.



### Strategy

Since we have a compound object in both cases, we can use the parallel-axis theorem to find the moment of inertia

about each axis. In (a), the center of mass of the sphere is located at a distance  $L + R$  from the axis of rotation. In (b), the center of mass of the sphere is located a distance  $R$  from the axis of rotation. In both cases, the moment of inertia of the rod is about an axis at one end. Refer to **Table 10.4** for the moments of inertia for the individual objects.

$$\text{a. } I_{\text{total}} = \sum_i I_i = I_{\text{Rod}} + I_{\text{Sphere}};$$

$$I_{\text{Sphere}} = I_{\text{center of mass}} + m_{\text{Sphere}}(L + R)^2 = \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}(L + R)^2;$$

$$I_{\text{total}} = I_{\text{Rod}} + I_{\text{Sphere}} = \frac{1}{3}m_{\text{Rod}}L^2 + \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}(L + R)^2;$$

$$I_{\text{total}} = \frac{1}{3}(2.0 \text{ kg})(0.5 \text{ m})^2 + \frac{2}{5}(1.0 \text{ kg})(0.2 \text{ m})^2 + (1.0 \text{ kg})(0.5 \text{ m} + 0.2 \text{ m})^2;$$

$$I_{\text{total}} = (0.167 + 0.016 + 0.490) \text{ kg} \cdot \text{m}^2 = 0.673 \text{ kg} \cdot \text{m}^2.$$

$$\text{b. } I_{\text{Sphere}} = \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}R^2;$$

$$I_{\text{total}} = I_{\text{Rod}} + I_{\text{Sphere}} = \frac{1}{3}m_{\text{Rod}}L^2 + \frac{2}{5}m_{\text{Sphere}}R^2 + m_{\text{Sphere}}R^2;$$

$$I_{\text{total}} = \frac{1}{3}(2.0 \text{ kg})(0.5 \text{ m})^2 + \frac{2}{5}(1.0 \text{ kg})(0.2 \text{ m})^2 + (1.0 \text{ kg})(0.2 \text{ m})^2;$$

$$I_{\text{total}} = (0.167 + 0.016 + 0.04) \text{ kg} \cdot \text{m}^2 = 0.223 \text{ kg} \cdot \text{m}^2.$$

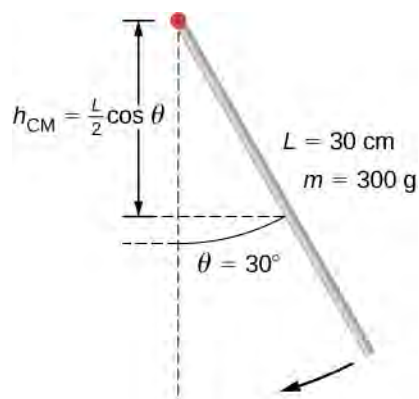
### Significance

Using the parallel-axis theorem eases the computation of the moment of inertia of compound objects. We see that the moment of inertia is greater in (a) than (b). This is because the axis of rotation is closer to the center of mass of the system in (b). The simple analogy is that of a rod. The moment of inertia about one end is  $\frac{1}{3}mL^2$ , but the moment of inertia through the center of mass along its length is  $\frac{1}{12}mL^2$ .

## Example 10.13

### Angular Velocity of a Pendulum

A pendulum in the shape of a rod (**Figure 10.30**) is released from rest at an angle of  $30^\circ$ . It has a length 30 cm and mass 300 g. What is its angular velocity at its lowest point?



**Figure 10.30** A pendulum in the form of a rod is released from rest at an angle of  $30^\circ$ .

**Strategy**

Use conservation of energy to solve the problem. At the point of release, the pendulum has gravitational potential energy, which is determined from the height of the center of mass above its lowest point in the swing. At the bottom of the swing, all of the gravitational potential energy is converted into rotational kinetic energy.

**Solution**

The change in potential energy is equal to the change in rotational kinetic energy,  $\Delta U + \Delta K = 0$ .

At the top of the swing:  $U = mgh_{\text{cm}} = mg\frac{L}{2}(\cos \theta)$ . At the bottom of the swing,  $U = mg\frac{L}{2}$ .

At the top of the swing, the rotational kinetic energy is  $K = 0$ . At the bottom of the swing,  $K = \frac{1}{2}I\omega^2$ .

Therefore:

$$\Delta U + \Delta K = 0 \Rightarrow (mg\frac{L}{2}(1 - \cos \theta) - 0) + (0 - \frac{1}{2}I\omega^2) = 0$$

or

$$\frac{1}{2}I\omega^2 = mg\frac{L}{2}(1 - \cos \theta).$$

Solving for  $\omega$ , we have

$$\omega = \sqrt{mg\frac{L}{I}(1 - \cos \theta)} = \sqrt{mg\frac{L}{\frac{1}{3}mL^2}(1 - \cos \theta)} = \sqrt{g\frac{3}{L}(1 - \cos \theta)}.$$

Inserting numerical values, we have

$$\omega = \sqrt{9.8 \text{ m/s}^2 \frac{3}{0.3 \text{ m}}(1 - \cos 30)} = 3.6 \text{ rad/s}.$$

**Significance**

Note that the angular velocity of the pendulum does not depend on its mass.

## 10.6 | Torque

### Learning Objectives

By the end of this section, you will be able to:

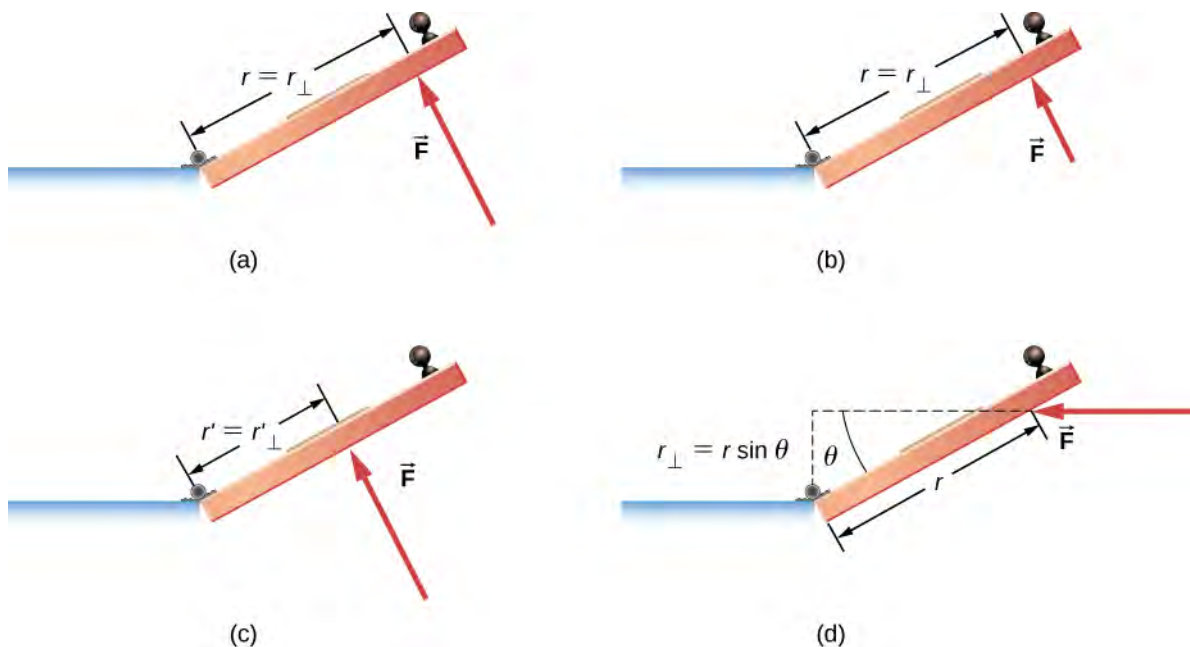
- Describe how the magnitude of a torque depends on the magnitude of the lever arm and the angle the force vector makes with the lever arm
- Determine the sign (positive or negative) of a torque using the right-hand rule
- Calculate individual torques about a common axis and sum them to find the net torque

An important quantity for describing the dynamics of a rotating rigid body is torque. We see the application of torque in many ways in our world. We all have an intuition about torque, as when we use a large wrench to unscrew a stubborn bolt. Torque is at work in unseen ways, as when we press on the accelerator in a car, causing the engine to put additional torque on the drive train. Or every time we move our bodies from a standing position, we apply a torque to our limbs. In this section, we define torque and make an argument for the equation for calculating torque for a rigid body with fixed-axis rotation.

### Defining Torque

So far we have defined many variables that are rotational equivalents to their translational counterparts. Let's consider what the counterpart to force must be. Since forces change the translational motion of objects, the rotational counterpart must be related to changing the rotational motion of an object about an axis. We call this rotational counterpart **torque**.

In everyday life, we rotate objects about an axis all the time, so intuitively we already know much about torque. Consider, for example, how we rotate a door to open it. First, we know that a door opens slowly if we push too close to its hinges; it is more efficient to rotate a door open if we push far from the hinges. Second, we know that we should push perpendicular to the plane of the door; if we push parallel to the plane of the door, we are not able to rotate it. Third, the larger the force, the more effective it is in opening the door; the harder you push, the more rapidly the door opens. The first point implies that the farther the force is applied from the axis of rotation, the greater the angular acceleration; the second implies that the effectiveness depends on the angle at which the force is applied; the third implies that the magnitude of the force must also be part of the equation. Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point. **Figure 10.31** shows counterclockwise rotations.



**Figure 10.31** Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) A counterclockwise torque is produced by a force  $\vec{F}$  acting at a distance  $r$  from the hinges (the pivot point). (b) A smaller counterclockwise torque is produced when a smaller force  $\vec{F}'$  acts at the same distance  $r$  from the hinges. (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) A smaller counterclockwise torque is produced by the same magnitude force as (a) acting at the same distance as (a) but at an angle  $\theta$  that is less than  $90^\circ$ .

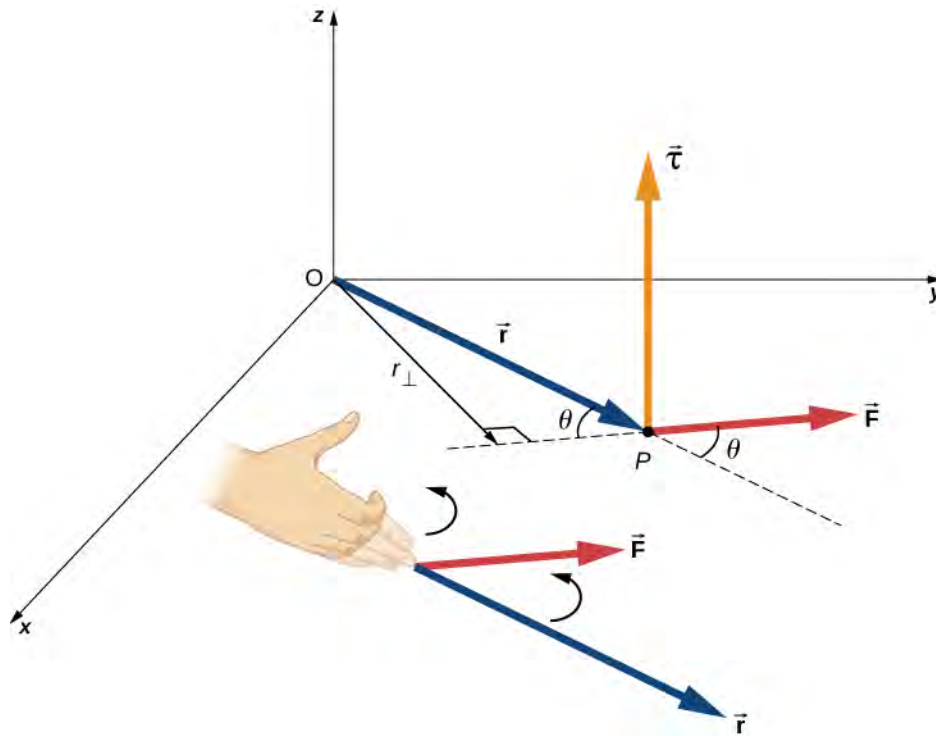
Now let's consider how to define torques in the general three-dimensional case.

### Torque

When a force  $\vec{F}$  is applied to a point  $P$  whose position is  $\vec{r}$  relative to  $O$  (**Figure 10.32**), the torque  $\vec{\tau}$  around  $O$  is

$$\vec{\tau} = \vec{r} \times \vec{F}. \quad (10.22)$$





**Figure 10.32** The torque is perpendicular to the plane defined by  $\vec{r}$  and  $\vec{F}$  and its direction is determined by the right-hand rule.

From the definition of the cross product, the torque  $\vec{\tau}$  is perpendicular to the plane containing  $\vec{r}$  and  $\vec{F}$  and has magnitude

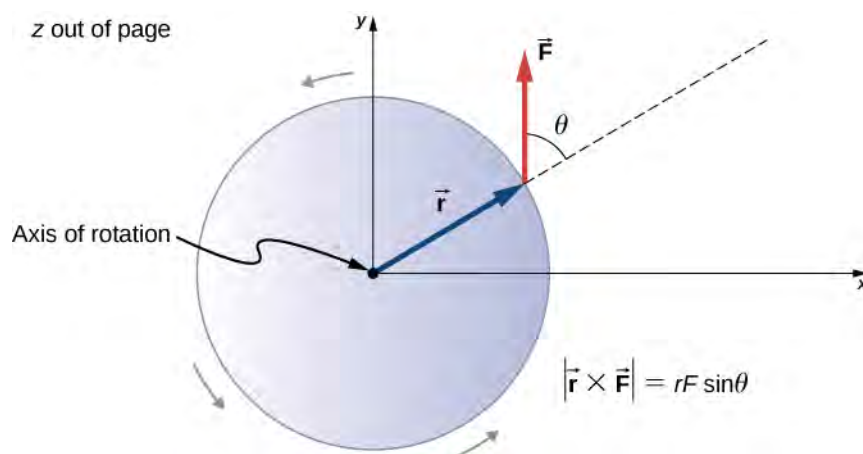
$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF \sin \theta,$$

where  $\theta$  is the angle between the vectors  $\vec{r}$  and  $\vec{F}$ . The SI unit of torque is newtons times meters, usually written as  $\text{N} \cdot \text{m}$ . The quantity  $r_{\perp} = r \sin \theta$  is the perpendicular distance from  $O$  to the line determined by the vector  $\vec{F}$  and is called the **lever arm**. Note that the greater the lever arm, the greater the magnitude of the torque. In terms of the lever arm, the magnitude of the torque is

$$|\vec{\tau}| = r_{\perp} F. \quad (10.23)$$

The cross product  $\vec{r} \times \vec{F}$  also tells us the sign of the torque. In **Figure 10.32**, the cross product  $\vec{r} \times \vec{F}$  is along the positive  $z$ -axis, which by convention is a positive torque. If  $\vec{r} \times \vec{F}$  is along the negative  $z$ -axis, this produces a negative torque.

If we consider a disk that is free to rotate about an axis through the center, as shown in **Figure 10.33**, we can see how the angle between the radius  $\vec{r}$  and the force  $\vec{F}$  affects the magnitude of the torque. If the angle is zero, the torque is zero; if the angle is  $90^\circ$ , the torque is maximum. The torque in **Figure 10.33** is positive because the direction of the torque by the right-hand rule is out of the page along the positive  $z$ -axis. The disk rotates counterclockwise due to the torque, in the same direction as a positive angular acceleration.



**Figure 10.33** A disk is free to rotate about its axis through the center. The magnitude of the torque on the disk is  $rF\sin\theta$ . When  $\theta = 0^\circ$ , the torque is zero and the disk does not rotate. When  $\theta = 90^\circ$ , the torque is maximum and the disk rotates with maximum angular acceleration.

Any number of torques can be calculated about a given axis. The individual torques add to produce a net torque about the axis. When the appropriate sign (positive or negative) is assigned to the magnitudes of individual torques about a specified axis, the net torque about the axis is the sum of the individual torques:

$$\vec{\tau}_{\text{net}} = \sum_i |\vec{\tau}_i|. \quad (10.24)$$

## Calculating Net Torque for Rigid Bodies on a Fixed Axis

In the following examples, we calculate the torque both abstractly and as applied to a rigid body.

We first introduce a problem-solving strategy.

### Problem-Solving Strategy: Finding Net Torque

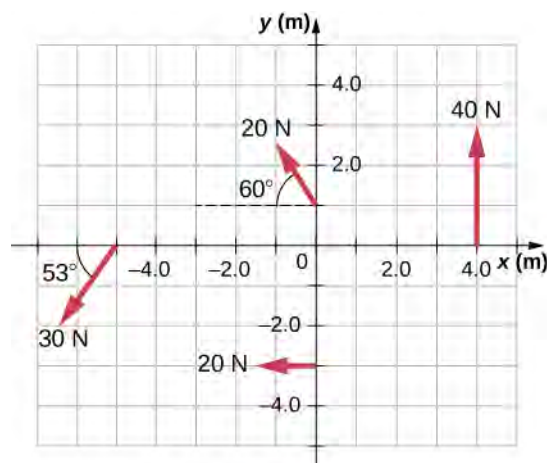
1. Choose a coordinate system with the pivot point or axis of rotation as the origin of the selected coordinate system.
2. Determine the angle between the lever arm  $\vec{r}$  and the force vector.
3. Take the cross product of  $\vec{r}$  and  $\vec{F}$  to determine if the torque is positive or negative about the pivot point or axis.
4. Evaluate the magnitude of the torque using  $r_\perp F$ .
5. Assign the appropriate sign, positive or negative, to the magnitude.
6. Sum the torques to find the net torque.

## Example 10.14

### Calculating Torque

Four forces are shown in **Figure 10.34** at particular locations and orientations with respect to a given xy-coordinate system. Find the torque due to each force about the origin, then use your results to find the net

torque about the origin.



**Figure 10.34** Four forces producing torques.

### Strategy

This problem requires calculating torque. All known quantities—forces with directions and lever arms—are given in the figure. The goal is to find each individual torque and the net torque by summing the individual torques. Be careful to assign the correct sign to each torque by using the cross product of  $\vec{r}$  and the force vector  $\vec{F}$ .

### Solution

Use  $|\vec{\tau}| = r_{\perp} F = rF\sin\theta$  to find the magnitude and  $\vec{\tau} = \vec{r} \times \vec{F}$  to determine the sign of the torque.

The torque from force 40 N in the first quadrant is given by  $(4)(40)\sin 90^{\circ} = 160 \text{ N} \cdot \text{m}$ .

The cross product of  $\vec{r}$  and  $\vec{F}$  is out of the page, positive.

The torque from force 20 N in the third quadrant is given by  $-(3)(20)\sin 90^{\circ} = -60 \text{ N} \cdot \text{m}$ .

The cross product of  $\vec{r}$  and  $\vec{F}$  is into the page, so it is negative.

The torque from force 30 N in the third quadrant is given by  $(5)(30)\sin 53^{\circ} = 120 \text{ N} \cdot \text{m}$ .

The cross product of  $\vec{r}$  and  $\vec{F}$  is out of the page, positive.

The torque from force 20 N in the second quadrant is given by  $(1)(20)\sin 30^{\circ} = 10 \text{ N} \cdot \text{m}$ .

The cross product of  $\vec{r}$  and  $\vec{F}$  is out of the page.

The net torque is therefore  $\tau_{\text{net}} = \sum_i |\tau_i| = 160 - 60 + 120 + 10 = 230 \text{ N} \cdot \text{m}$ .

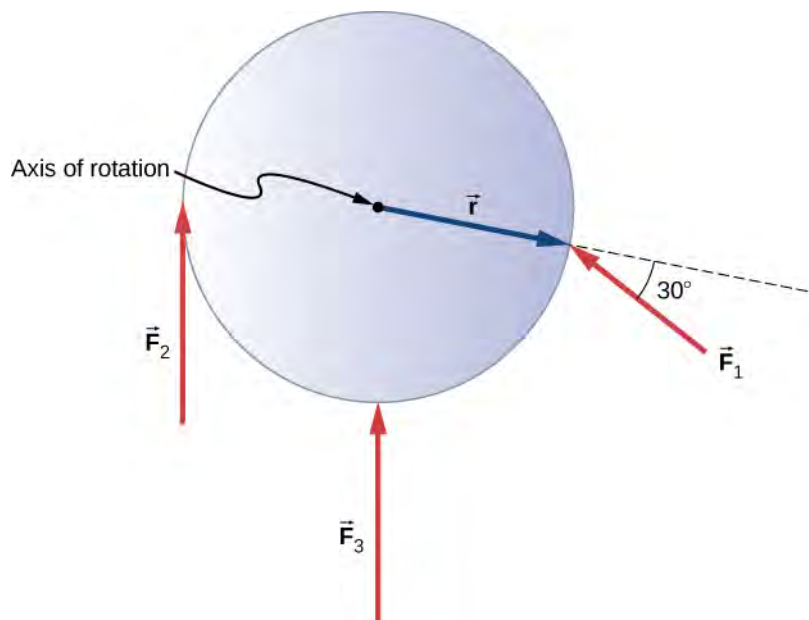
### Significance

Note that each force that acts in the counterclockwise direction has a positive torque, whereas each force that acts in the clockwise direction has a negative torque. The torque is greater when the distance, force, or perpendicular components are greater.

## Example 10.15

### Calculating Torque on a rigid body

**Figure 10.35** shows several forces acting at different locations and angles on a flywheel. We have  $|\vec{F}_1| = 20\text{ N}$ ,  $|\vec{F}_2| = 30\text{ N}$ ,  $|\vec{F}_3| = 30\text{ N}$ , and  $r = 0.5\text{ m}$ . Find the net torque on the flywheel about an axis through the center.



**Figure 10.35** Three forces acting on a flywheel.

### Strategy

We calculate each torque individually, using the cross product, and determine the sign of the torque. Then we sum the torques to find the net torque.

### Solution

We start with  $\vec{F}_1$ . If we look at **Figure 10.35**, we see that  $\vec{F}_1$  makes an angle of  $90^\circ + 60^\circ$  with the radius vector  $\vec{r}$ . Taking the cross product, we see that it is out of the page and so is positive. We also see this from calculating its magnitude:

$$|\vec{\tau}_1| = rF_1 \sin 150^\circ = 0.5\text{ m}(20\text{ N})(0.5) = 5.0\text{ N}\cdot\text{m}.$$

Next we look at  $\vec{F}_2$ . The angle between  $\vec{F}_2$  and  $\vec{r}$  is  $90^\circ$  and the cross product is into the page so the torque is negative. Its value is

$$|\vec{\tau}_2| = -rF_2 \sin 90^\circ = -0.5\text{ m}(30\text{ N}) = -15.0\text{ N}\cdot\text{m}.$$

When we evaluate the torque due to  $\vec{F}_3$ , we see that the angle it makes with  $\vec{r}$  is zero so  $\vec{r} \times \vec{F}_3 = 0$ .


Therefore,  $\vec{F}_3$  does not produce any torque on the flywheel.

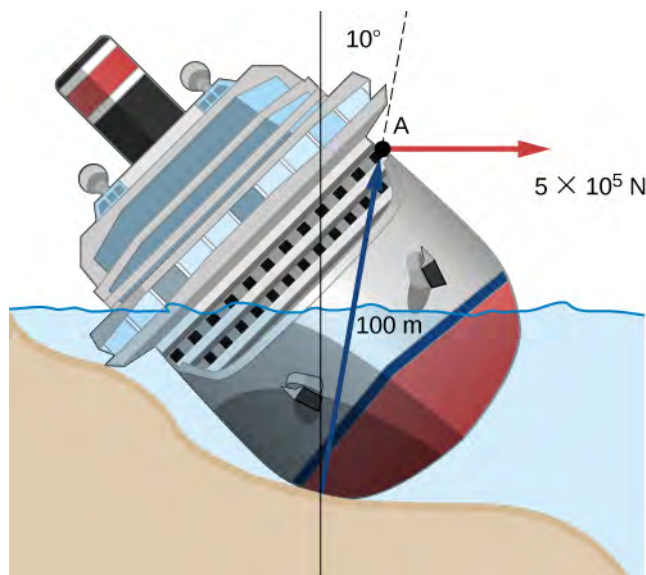
We evaluate the sum of the torques:

$$\tau_{\text{net}} = \sum_i |\tau_i| = 5 - 15 = -10\text{ N}\cdot\text{m}.$$

### Significance

The axis of rotation is at the center of mass of the flywheel. Since the flywheel is on a fixed axis, it is not free to translate. If it were on a frictionless surface and not fixed in place,  $\vec{F}_3$  would cause the flywheel to translate, as well as  $\vec{F}_1$ . Its motion would be a combination of translation and rotation.

-  **10.6 Check Your Understanding** A large ocean-going ship runs aground near the coastline, similar to the fate of the *Costa Concordia*, and lies at an angle as shown below. Salvage crews must apply a torque to right the ship in order to float the vessel for transport. A force of  $5.0 \times 10^5$  N acting at point A must be applied to right the ship. What is the torque about the point of contact of the ship with the ground (**Figure 10.36**)?



**Figure 10.36** A ship runs aground and tilts, requiring torque to be applied to return the vessel to an upright position.

## 10.7 | Newton's Second Law for Rotation

### Learning Objectives

By the end of this section, you will be able to:

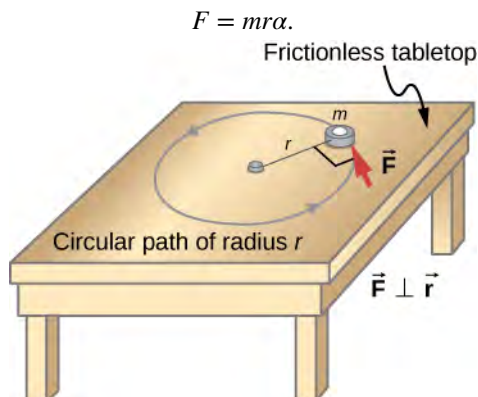
- Calculate the torques on rotating systems about a fixed axis to find the angular acceleration
- Explain how changes in the moment of inertia of a rotating system affect angular acceleration with a fixed applied torque

In this section, we put together all the pieces learned so far in this chapter to analyze the dynamics of rotating rigid bodies. We have analyzed motion with kinematics and rotational kinetic energy but have not yet connected these ideas with force and/or torque. In this section, we introduce the rotational equivalent to Newton's second law of motion and apply it to rigid bodies with fixed-axis rotation.

### Newton's Second Law for Rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton's second law,  $\Sigma \vec{F} = m \vec{a}$ , which involves torque and rotational motion? To investigate this, we start with Newton's second law for

a single particle rotating around an axis and executing circular motion. Let's exert a force  $\vec{F}$  on a point mass  $m$  that is at a distance  $r$  from a pivot point (**Figure 10.37**). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton's second law to determine the magnitude of the acceleration  $a = F/m$  in the direction of  $\vec{F}$ . Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by  $a = r\alpha$ . Substituting this expression into Newton's second law, we obtain



**Figure 10.37** An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force  $\vec{F}$  is applied to the object perpendicular to the radius  $r$ , causing it to accelerate about the pivot point. The force is perpendicular to  $r$ .

Multiply both sides of this equation by  $r$ ,

$$rF = mr^2\alpha.$$

Note that the left side of this equation is the torque about the axis of rotation, where  $r$  is the lever arm and  $F$  is the force, perpendicular to  $r$ . Recall that the moment of inertia for a point particle is  $I = mr^2$ . The torque applied perpendicularly to the point mass in **Figure 10.37** is therefore

$$\tau = I\alpha.$$

*The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.*

### Newton's Second Law for Rotation

If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\alpha. \quad (10.25)$$

The term  $I\alpha$  is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

**Equation 10.25** is **Newton's second law for rotation** and tells us how to relate torque, moment of inertia, and rotational kinematics. This is called the equation for **rotational dynamics**. With this equation, we can solve a whole class of problems involving force and rotation. It makes sense that the relationship for how much force it takes to rotate a body would include the moment of inertia, since that is the quantity that tells us how easy or hard it is to change the rotational motion of an object.

## Deriving Newton's Second Law for Rotation in Vector Form

As before, when we found the angular acceleration, we may also find the torque vector. The second law  $\Sigma \vec{F} = m \vec{a}$  tells us the relationship between net force and how to change the translational motion of an object. We have a vector rotational equivalent of this equation, which can be found by using **Equation 10.7** and **Figure 10.8**. **Equation 10.7** relates the angular acceleration to the position and tangential acceleration vectors:

$$\vec{a} = \vec{\alpha} \times \vec{r}.$$

We form the cross product of this equation with  $\vec{r}$  and use a cross product identity (note that  $\vec{r} \cdot \vec{\alpha} = 0$ ):

$$\vec{r} \times \vec{a} = \vec{r} \times (\vec{\alpha} \times \vec{r}) = \vec{\alpha} (\vec{r} \cdot \vec{r}) - \vec{r} (\vec{r} \cdot \vec{\alpha}) = \vec{\alpha} (\vec{r} \cdot \vec{r}) = \vec{\alpha} r^2.$$

We now form the cross product of Newton's second law with the position vector  $\vec{r}$ ,

$$\Sigma (\vec{r} \times \vec{F}) = \vec{r} \times (m \vec{a}) = m \vec{r} \times \vec{a} = m r^2 \vec{\alpha}.$$

Identifying the first term on the left as the sum of the torques, and  $m r^2$  as the moment of inertia, we arrive at Newton's second law of rotation in vector form:

$$\Sigma \vec{\tau} = I \vec{\alpha}. \quad (10.26)$$

This equation is exactly **Equation 10.25** but with the torque and angular acceleration as vectors. An important point is that the torque vector is in the same direction as the angular acceleration.

## Applying the Rotational Dynamics Equation

Before we apply the rotational dynamics equation to some everyday situations, let's review a general problem-solving strategy for use with this category of problems.

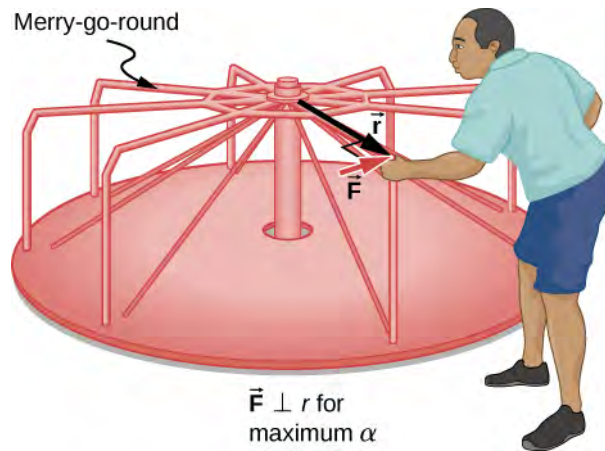
### Problem-Solving Strategy: Rotational Dynamics

1. Examine the situation to determine that torque and mass are involved in the rotation. Draw a careful sketch of the situation.
2. Determine the system of interest.
3. Draw a free-body diagram. That is, draw and label all external forces acting on the system of interest.
4. Identify the pivot point. If the object is in equilibrium, it must be in equilibrium for all possible pivot points—choose the one that simplifies your work the most.
5. Apply  $\sum_i \tau_i = I \alpha$ , the rotational equivalent of Newton's second law, to solve the problem. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
6. As always, check the solution to see if it is reasonable.

## Example 10.16

### Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in **Figure 10.38**. He exerts a force of 250 N at the edge of the 200.0-kg merry-go-round, which has a 1.50-m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible friction.



**Figure 10.38** A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

### Strategy

The net torque is given directly by the expression  $\sum_i \tau_i = I\alpha$ . To solve for  $\alpha$ , we must first calculate the net torque  $\tau$  (which is the same in both cases) and moment of inertia  $I$  (which is greater in the second case).

### Solution

- a. The moment of inertia of a solid disk about this axis is given in **Figure 10.20** to be

$$\frac{1}{2}MR^2.$$

We have  $M = 50.0 \text{ kg}$  and  $R = 1.50 \text{ m}$ , so

$$I = (0.500)(50.0 \text{ kg})(1.50 \text{ m})^2 = 56.25 \text{ kg}\cdot\text{m}^2.$$

To find the net torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau = rF\sin\theta = (1.50 \text{ m})(250.0 \text{ N}) = 375.0 \text{ N}\cdot\text{m}.$$

Now, after we substitute the known values, we find the angular acceleration to be

$$\alpha = \frac{\tau}{I} = \frac{375.0 \text{ N}\cdot\text{m}}{56.25 \text{ kg}\cdot\text{m}^2} = 6.67 \frac{\text{rad}}{\text{s}^2}.$$

- b. We expect the angular acceleration for the system to be less in this part because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia  $I$ , we first find the child's moment of inertia  $I_c$  by approximating the child as a point mass at a distance of 1.25 m from the axis. Then

$$I_c = mR^2 = (18.0 \text{ kg})(1.25 \text{ m})^2 = 28.13 \text{ kg}\cdot\text{m}^2.$$

The total moment of inertia is the sum of the moments of inertia of the merry-go-round and the child (about the same axis):

$$I = 28.13 \text{ kg}\cdot\text{m}^2 + 56.25 \text{ kg}\cdot\text{m}^2 = 84.38 \text{ kg}\cdot\text{m}^2.$$

Substituting known values into the equation for  $\alpha$  gives



$$\alpha = \frac{\tau}{I} = \frac{375.0 \text{ N}\cdot\text{m}}{84.38 \text{ kg}\cdot\text{m}^2} = 4.44 \frac{\text{rad}}{\text{s}^2}.$$

### Significance

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case.



**10.7 Check Your Understanding** The fan blades on a jet engine have a moment of inertia  $30.0 \text{ kg}\cdot\text{m}^2$ . In 10 s, they rotate counterclockwise from rest up to a rotation rate of 20 rev/s. (a) What torque must be applied to the blades to achieve this angular acceleration? (b) What is the torque required to bring the fan blades rotating at 20 rev/s to a rest in 20 s?

## 10.8 | Work and Power for Rotational Motion

### Learning Objectives

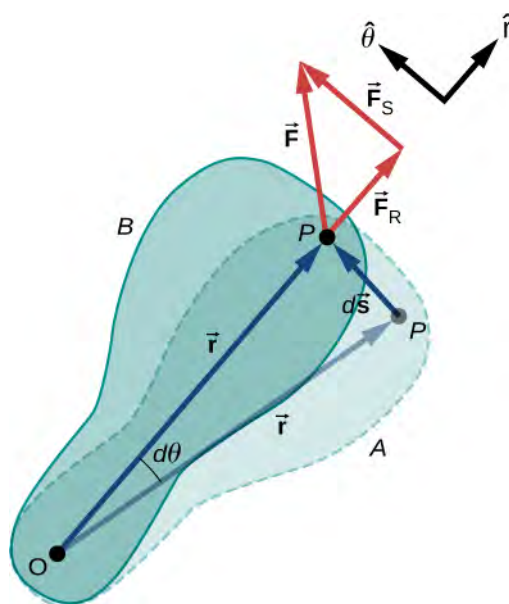
By the end of this section, you will be able to:

- Use the work-energy theorem to analyze rotation to find the work done on a system when it is rotated about a fixed axis for a finite angular displacement
- Solve for the angular velocity of a rotating rigid body using the work-energy theorem
- Find the power delivered to a rotating rigid body given the applied torque and angular velocity
- Summarize the rotational variables and equations and relate them to their translational counterparts

Thus far in the chapter, we have extensively addressed kinematics and dynamics for rotating rigid bodies around a fixed axis. In this final section, we define work and power within the context of rotation about a fixed axis, which has applications to both physics and engineering. The discussion of work and power makes our treatment of rotational motion almost complete, with the exception of rolling motion and angular momentum, which are discussed in **Angular Momentum**. We begin this section with a treatment of the work-energy theorem for rotation.

### Work for Rotational Motion

Now that we have determined how to calculate kinetic energy for rotating rigid bodies, we can proceed with a discussion of the work done on a rigid body rotating about a fixed axis. **Figure 10.39** shows a rigid body that has rotated through an angle  $d\theta$  from  $A$  to  $B$  while under the influence of a force  $\vec{F}$ . The external force  $\vec{F}$  is applied to point  $P$ , whose position is  $\vec{r}$ , and the rigid body is constrained to rotate about a fixed axis that is perpendicular to the page and passes through  $O$ . The rotational axis is fixed, so the vector  $\vec{r}$  moves in a circle of radius  $r$ , and the vector  $d\vec{s}$  is perpendicular to  $\vec{r}$ .



**Figure 10.39** A rigid body rotates through an angle  $d\theta$  from A to B by the action of an external force  $\vec{F}$  applied to point P.

From **Equation 10.2**, we have

$$\vec{s} = \vec{\theta} \times \vec{r}.$$

Thus,

$$d\vec{s} = d(\vec{\theta} \times \vec{r}) = d\vec{\theta} \times \vec{r} + d\vec{r} \times \vec{\theta} = d\vec{\theta} \times \vec{r}.$$

Note that  $d\vec{r}$  is zero because  $\vec{r}$  is fixed on the rigid body from the origin O to point P. Using the definition of work, we obtain

$$W = \int \sum \vec{F} \cdot d\vec{s} = \int \sum \vec{F} \cdot (d\vec{\theta} \times \vec{r}) = \int d\vec{\theta} \cdot (\vec{r} \times \sum \vec{F})$$

where we used the identity  $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$ . Noting that  $(\vec{r} \times \sum \vec{F}) = \sum \vec{\tau}$ , we arrive at the expression for the **rotational work** done on a rigid body:

$$W = \int \sum \vec{\tau} \cdot d\vec{\theta}. \quad (10.27)$$

The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates. The incremental work is

$$dW = \left( \sum_i \tau_i \right) d\theta \quad (10.28)$$

where we have taken the dot product in **Equation 10.27**, leaving only torques along the axis of rotation. In a rigid body, all particles rotate through the same angle; thus the work of every external force is equal to the torque times the common incremental angle  $d\theta$ . The quantity  $\left( \sum_i \tau_i \right)$  is the net torque on the body due to external forces.

Similarly, we found the kinetic energy of a rigid body rotating around a fixed axis by summing the kinetic energy of each particle that makes up the rigid body. Since the work-energy theorem  $W_i = \Delta K_i$  is valid for each particle, it is valid for the

sum of the particles and the entire body.

### Work-Energy Theorem for Rotation

The work-energy theorem for a rigid body rotating around a fixed axis is

$$W_{AB} = K_B - K_A \quad (10.29)$$

where

$$K = \frac{1}{2}I\omega^2$$

and the rotational work done by a net force rotating a body from point  $A$  to point  $B$  is

$$W_{AB} = \int_{\theta_A}^{\theta_B} \left( \sum_i \tau_i \right) d\theta. \quad (10.30)$$

We give a strategy for using this equation when analyzing rotational motion.

### Problem-Solving Strategy: Work-Energy Theorem for Rotational Motion

1. Identify the forces on the body and draw a free-body diagram. Calculate the torque for each force.
2. Calculate the work done during the body's rotation by every torque.
3. Apply the work-energy theorem by equating the net work done on the body to the change in rotational kinetic energy.

Let's look at two examples and use the work-energy theorem to analyze rotational motion.

## Example 10.17

### Rotational Work and Energy

A  $12.0 \text{ N} \cdot \text{m}$  torque is applied to a flywheel that rotates about a fixed axis and has a moment of inertia of  $30.0 \text{ kg} \cdot \text{m}^2$ . If the flywheel is initially at rest, what is its angular velocity after it has turned through eight revolutions?

#### Strategy

We apply the work-energy theorem. We know from the problem description what the torque is and the angular displacement of the flywheel. Then we can solve for the final angular velocity.

#### Solution

The flywheel turns through eight revolutions, which is  $16\pi$  radians. The work done by the torque, which is constant and therefore can come outside the integral in **Equation 10.30**, is

$$W_{AB} = \tau(\theta_B - \theta_A).$$

We apply the work-energy theorem:

$$W_{AB} = \tau(\theta_B - \theta_A) = \frac{1}{2}I\omega_B^2 - \frac{1}{2}I\omega_A^2.$$

With  $\tau = 12.0 \text{ N} \cdot \text{m}$ ,  $\theta_B - \theta_A = 16.0\pi \text{ rad}$ ,  $I = 30.0 \text{ kg} \cdot \text{m}^2$ , and  $\omega_A = 0$ , we have

$$12.0 \text{ N} \cdot \text{m}(16.0\pi \text{ rad}) = \frac{1}{2}(30.0 \text{ kg} \cdot \text{m}^2)(\omega_B^2) - 0.$$

Therefore,

$$\omega_B = 6.3 \text{ rad/s.}$$

This is the angular velocity of the flywheel after eight revolutions.

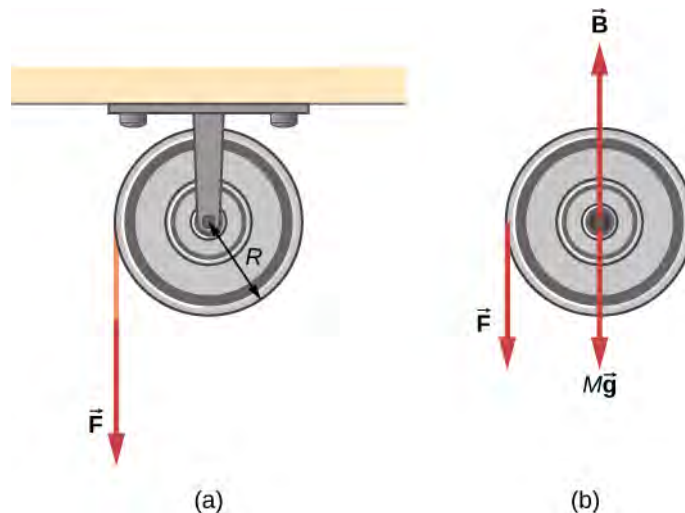
### Significance

The work-energy theorem provides an efficient way to analyze rotational motion, connecting torque with rotational kinetic energy.

## Example 10.18

### Rotational Work: A Pulley

A string wrapped around the pulley in **Figure 10.40** is pulled with a constant downward force  $\vec{F}$  of magnitude 50 N. The radius  $R$  and moment of inertia  $I$  of the pulley are 0.10 m and  $2.5 \times 10^{-3} \text{ kg}\cdot\text{m}^2$ , respectively. If the string does not slip, what is the angular velocity of the pulley after 1.0 m of string has unwound? Assume the pulley starts from rest.



**Figure 10.40** (a) A string is wrapped around a pulley of radius  $R$ .  
(b) The free-body diagram.

### Strategy

Looking at the free-body diagram, we see that neither  $\vec{B}$ , the force on the bearings of the pulley, nor  $M\vec{g}$ , the weight of the pulley, exerts a torque around the rotational axis, and therefore does no work on the pulley. As the pulley rotates through an angle  $\theta$ ,  $\vec{F}$  acts through a distance  $d$  such that  $d = R\theta$ .

### Solution

Since the torque due to  $\vec{F}$  has magnitude  $\tau = RF$ , we have

$$W = \tau\theta = (FR)\theta = Fd.$$

If the force on the string acts through a distance of 1.0 m, we have, from the work-energy theorem,

$$\begin{aligned}
 W_{AB} &= K_B - K_A \\
 Fd &= \frac{1}{2}I\omega^2 - 0 \\
 (50.0 \text{ N})(1.0 \text{ m}) &= \frac{1}{2}(2.5 \times 10^{-3} \text{ kg}\cdot\text{m}^2)\omega^2.
 \end{aligned}$$

Solving for  $\omega$ , we obtain

$$\omega = 200.0 \text{ rad/s.}$$

## Power for Rotational Motion

Power always comes up in the discussion of applications in engineering and physics. Power for rotational motion is equally as important as power in linear motion and can be derived in a similar way as in linear motion when the force is a constant.

The linear power when the force is a constant is  $P = \vec{F} \cdot \vec{v}$ . If the net torque is constant over the angular displacement, **Equation 10.25** simplifies and the net torque can be taken out of the integral. In the following discussion, we assume the net torque is constant. We can apply the definition of power derived in **Power** to rotational motion. From **Work and Kinetic Energy**, the instantaneous power (or just power) is defined as the rate of doing work,

$$P = \frac{dW}{dt}.$$

If we have a constant net torque, **Equation 10.25** becomes  $W = \tau\theta$  and the power is

$$P = \frac{dW}{dt} = \frac{d}{dt}(\tau\theta) = \tau \frac{d\theta}{dt}$$

or

$$P = \tau\omega. \quad (10.31)$$

### Example 10.19

#### Torque on a Boat Propeller

A boat engine operating at  $9.0 \times 10^4 \text{ W}$  is running at 300 rev/min. What is the torque on the propeller shaft?

#### Strategy

We are given the rotation rate in rev/min and the power consumption, so we can easily calculate the torque.

#### Solution

$$\begin{aligned}
 300.0 \text{ rev/min} &= 31.4 \text{ rad/s;} \\
 \tau &= \frac{P}{\omega} = \frac{9.0 \times 10^4 \text{ N}\cdot\text{m/s}}{31.4 \text{ rad/s}} = 2864.8 \text{ N}\cdot\text{m}.
 \end{aligned}$$

#### Significance

It is important to note the radian is a dimensionless unit because its definition is the ratio of two lengths. It therefore does not appear in the solution.



**10.8 Check Your Understanding** A constant torque of  $500 \text{ kN}\cdot\text{m}$  is applied to a wind turbine to keep it rotating at  $6 \text{ rad/s}$ . What is the power required to keep the turbine rotating?

## Rotational and Translational Relationships Summarized

The rotational quantities and their linear analog are summarized in three tables. **Table 10.5** summarizes the rotational variables for circular motion about a fixed axis with their linear analogs and the connecting equation, except for the

centripetal acceleration, which stands by itself. **Table 10.6** summarizes the rotational and translational kinematic equations. **Table 10.7** summarizes the rotational dynamics equations with their linear analogs.

Rotational	Translational	Relationship
$\theta$	$x$	$\theta = \frac{s}{r}$
$\omega$	$v_t$	$\omega = \frac{v_t}{r}$
$\alpha$	$a_t$	$\alpha = \frac{a_t}{r}$
	$a_c$	$a_c = \frac{v_t^2}{r}$

**Table 10.5 Rotational and Translational Variables: Summary**

Rotational	Translational
$\theta_f = \theta_0 + \bar{\omega} t$	$x = x_0 + \bar{v} t$
$\omega_f = \omega_0 + \alpha t$	$v_f = v_0 + at$
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_0^2 + 2a(\Delta x)$

**Table 10.6 Rotational and Translational Kinematic Equations: Summary**

Rotational	Translational
$I = \sum_i m_i r_i^2$	$m$
$K = \frac{1}{2}I\omega^2$	$K = \frac{1}{2}mv^2$
$\sum_i \tau_i = I\alpha$	$\sum_i \vec{F}_i = m \vec{a}$
$W_{AB} = \int_{\theta_A}^{\theta_B} \left( \sum_i \tau_i \right) d\theta$	$W = \int \vec{F} \cdot d\vec{s}$
$P = \tau\omega$	$P = \vec{F} \cdot \vec{v}$

**Table 10.7 Rotational and Translational Equations: Dynamics**

# CHAPTER 10 REVIEW

## KEY TERMS

**angular acceleration** time rate of change of angular velocity

**angular position** angle a body has rotated through in a fixed coordinate system

**angular velocity** time rate of change of angular position

**instantaneous angular acceleration** derivative of angular velocity with respect to time

**instantaneous angular velocity** derivative of angular position with respect to time

**kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time

**lever arm** perpendicular distance from the line that the force vector lies on to a given axis

**linear mass density** the mass per unit length  $\lambda$  of a one dimensional object

**moment of inertia** rotational mass of rigid bodies that relates to how easy or hard it will be to change the angular velocity of the rotating rigid body

**Newton's second law for rotation** sum of the torques on a rotating system equals its moment of inertia times its angular acceleration

**parallel axis** axis of rotation that is parallel to an axis about which the moment of inertia of an object is known

**parallel-axis theorem** if the moment of inertia is known for a given axis, it can be found for any axis parallel to it

**rotational dynamics** analysis of rotational motion using the net torque and moment of inertia to find the angular acceleration

**rotational kinetic energy** kinetic energy due to the rotation of an object; this is part of its total kinetic energy

**rotational work** work done on a rigid body due to the sum of the torques integrated over the angle through which the body rotates

**surface mass density** mass per unit area  $\sigma$  of a two dimensional object

**torque** cross product of a force and a lever arm to a given axis

**total linear acceleration** vector sum of the centripetal acceleration vector and the tangential acceleration vector

**work-energy theorem for rotation** the total rotational work done on a rigid body is equal to the change in rotational kinetic energy of the body

## KEY EQUATIONS

Angular position

$$\theta = \frac{s}{r}$$

Angular velocity

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta \theta}{\Delta t} = \frac{d\theta}{dt}$$

Tangential speed

$$v_t = r\omega$$

Angular acceleration

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

Tangential acceleration

$$a_t = r\alpha$$

Average angular velocity

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2}$$

Angular displacement

$$\theta_f = \theta_0 + \bar{\omega} t$$

Angular velocity from constant angular acceleration	$\omega_f = \omega_0 + \alpha t$
Angular velocity from displacement and constant angular acceleration	$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
Change in angular velocity	$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$
Total acceleration	$\vec{\mathbf{a}} = \vec{\mathbf{a}}_c + \vec{\mathbf{a}}_t$
Rotational kinetic energy	$K = \frac{1}{2}\left(\sum_j m_j r_j^2\right)\omega^2$
Moment of inertia	$I = \sum_j m_j r_j^2$
Rotational kinetic energy in terms of the moment of inertia of a rigid body	$K = \frac{1}{2}I\omega^2$
Moment of inertia of a continuous object	$I = \int r^2 dm$
Parallel-axis theorem	$I_{\text{parallel-axis}} = I_{\text{initial}} + md^2$
Moment of inertia of a compound object	$I_{\text{total}} = \sum_i I_i$
Torque vector	$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$
Magnitude of torque	$ \vec{\boldsymbol{\tau}}  = r_{\perp} F$
Total torque	$\tau_{\text{net}} = \sum_i  \tau_i $
Newton's second law for rotation	$\sum_i \tau_i = I\alpha$
Incremental work done by a torque	$dW = \left(\sum_i \tau_i\right)d\theta$
Work-energy theorem	$W_{AB} = K_B - K_A$
Rotational work done by net force	$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right)d\theta$
Rotational power	$P = \tau\omega$

## SUMMARY

### 10.1 Rotational Variables

- The angular position  $\theta$  of a rotating body is the angle the body has rotated through in a fixed coordinate system, which serves as a frame of reference.
- The angular velocity of a rotating body about a fixed axis is defined as  $\omega(\text{rad/s})$ , the rotational rate of the body in radians per second. The instantaneous angular velocity of a rotating body  $\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$  is the derivative



with respect to time of the angular position  $\theta$ , found by taking the limit  $\Delta t \rightarrow 0$  in the average angular velocity  $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$ . The angular velocity relates  $v_t$  to the tangential speed of a point on the rotating body through the relation  $v_t = r\omega$ , where  $r$  is the radius to the point and  $v_t$  is the tangential speed at the given point.

- The angular velocity  $\vec{\omega}$  is found using the right-hand rule. If the fingers curl in the direction of rotation about a fixed axis, the thumb points in the direction of  $\vec{\omega}$  (see **Figure 10.5**).
- If the system's angular velocity is not constant, then the system has an angular acceleration. The average angular acceleration over a given time interval is the change in angular velocity over this time interval,  $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ . The instantaneous angular acceleration is the time derivative of angular velocity,  $\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$ . The angular acceleration  $\vec{\alpha}$  is found by locating the angular velocity. If a rotation rate of a rotating body is decreasing, the angular acceleration is in the opposite direction to  $\vec{\omega}$ . If the rotation rate is increasing, the angular acceleration is in the same direction as  $\vec{\omega}$ .
- The tangential acceleration of a point at a radius from the axis of rotation is the angular acceleration times the radius to the point.

## 10.2 Rotation with Constant Angular Acceleration

- The kinematics of rotational motion describes the relationships among rotation angle (angular position), angular velocity, angular acceleration, and time.
- For a constant angular acceleration, the angular velocity varies linearly. Therefore, the average angular velocity is  $1/2$  the initial plus final angular velocity over a given time period:

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2}.$$

- We used a graphical analysis to find solutions to fixed-axis rotation with constant angular acceleration. From the relation  $\omega = \frac{d\theta}{dt}$ , we found that the area under an angular velocity-vs.-time curve gives the angular displacement,

$\theta_f - \theta_0 = \Delta\theta = \int_{t_0}^t \omega(t)dt$ . The results of the graphical analysis were verified using the kinematic equations for

constant angular acceleration. Similarly, since  $\alpha = \frac{d\omega}{dt}$ , the area under an angular acceleration-vs.-time graph gives

the change in angular velocity:  $\omega_f - \omega_0 = \Delta\omega = \int_{t_0}^t \alpha(t)dt$ .

## 10.3 Relating Angular and Translational Quantities

- The linear kinematic equations have their rotational counterparts such that there is a mapping  $x \rightarrow \theta$ ,  $v \rightarrow \omega$ ,  $a \rightarrow \alpha$ .
- A system undergoing uniform circular motion has a constant angular velocity, but points at a distance  $r$  from the rotation axis have a linear centripetal acceleration.
- A system undergoing nonuniform circular motion has an angular acceleration and therefore has both a linear centripetal and linear tangential acceleration at a point a distance  $r$  from the axis of rotation.
- The total linear acceleration is the vector sum of the centripetal acceleration vector and the tangential acceleration vector. Since the centripetal and tangential acceleration vectors are perpendicular to each other for circular motion, the magnitude of the total linear acceleration is  $|\vec{\mathbf{a}}| = \sqrt{a_c^2 + a_t^2}$ .

### 10.4 Moment of Inertia and Rotational Kinetic Energy

- The rotational kinetic energy is the kinetic energy of rotation of a rotating rigid body or system of particles, and is given by  $K = \frac{1}{2}I\omega^2$ , where  $I$  is the moment of inertia, or “rotational mass” of the rigid body or system of particles.
- The moment of inertia for a system of point particles rotating about a fixed axis is  $I = \sum_j m_j r_j^2$ , where  $m_j$  is the mass of the point particle and  $r_j$  is the distance of the point particle to the rotation axis. Because of the  $r^2$  term, the moment of inertia increases as the square of the distance to the fixed rotational axis. The moment of inertia is the rotational counterpart to the mass in linear motion.
- In systems that are both rotating and translating, conservation of mechanical energy can be used if there are no nonconservative forces at work. The total mechanical energy is then conserved and is the sum of the rotational and translational kinetic energies, and the gravitational potential energy.

### 10.5 Calculating Moments of Inertia

- Moments of inertia can be found by summing or integrating over every ‘piece of mass’ that makes up an object, multiplied by the square of the distance of each ‘piece of mass’ to the axis. In integral form the moment of inertia is  $I = \int r^2 dm$ .
- Moment of inertia is larger when an object’s mass is farther from the axis of rotation.
- It is possible to find the moment of inertia of an object about a new axis of rotation once it is known for a parallel axis. This is called the parallel axis theorem given by  $I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2$ , where  $d$  is the distance from the initial axis to the parallel axis.
- Moment of inertia for a compound object is simply the sum of the moments of inertia for each individual object that makes up the compound object.

### 10.6 Torque

- The magnitude of a torque about a fixed axis is calculated by finding the lever arm to the point where the force is applied and using the relation  $|\vec{\tau}| = r_{\perp} F$ , where  $r_{\perp}$  is the perpendicular distance from the axis to the line upon which the force vector lies.
- The sign of the torque is found using the right hand rule. If the page is the plane containing  $\vec{r}$  and  $\vec{F}$ , then  $\vec{r} \times \vec{F}$  is out of the page for positive torques and into the page for negative torques.
- The net torque can be found from summing the individual torques about a given axis.

### 10.7 Newton’s Second Law for Rotation

- Newton’s second law for rotation,  $\sum_i \tau_i = I\alpha$ , says that the sum of the torques on a rotating system about a fixed axis equals the product of the moment of inertia and the angular acceleration. This is the rotational analog to Newton’s second law of linear motion.
- In the vector form of Newton’s second law for rotation, the torque vector  $\vec{\tau}$  is in the same direction as the angular acceleration  $\vec{\alpha}$ . If the angular acceleration of a rotating system is positive, the torque on the system is also positive, and if the angular acceleration is negative, the torque is negative.

### 10.8 Work and Power for Rotational Motion

- The incremental work  $dW$  in rotating a rigid body about a fixed axis is the sum of the torques about the axis times the incremental angle  $d\theta$ .

- The total work done to rotate a rigid body through an angle  $\theta$  about a fixed axis is the sum of the torques integrated over the angular displacement. If the torque is a constant as a function of  $\theta$ , then  $W_{AB} = \tau(\theta_B - \theta_A)$ .
- The work-energy theorem relates the rotational work done to the change in rotational kinetic energy:  $W_{AB} = K_B - K_A$  where  $K = \frac{1}{2}I\omega^2$ .
- The power delivered to a system that is rotating about a fixed axis is the torque times the angular velocity,  $P = \tau\omega$ .

## CONCEPTUAL QUESTIONS

### 10.1 Rotational Variables

1. A clock is mounted on the wall. As you look at it, what is the direction of the angular velocity vector of the second hand?
2. What is the value of the angular acceleration of the second hand of the clock on the wall?
3. A baseball bat is swung. Do all points on the bat have the same angular velocity? The same tangential speed?
4. The blades of a blender on a counter are rotating clockwise as you look into it from the top. If the blender is put to a greater speed what direction is the angular acceleration of the blades?

### 10.2 Rotation with Constant Angular Acceleration

5. If a rigid body has a constant angular acceleration, what is the functional form of the angular velocity in terms of the time variable?
6. If a rigid body has a constant angular acceleration, what is the functional form of the angular position?
7. If the angular acceleration of a rigid body is zero, what is the functional form of the angular velocity?
8. A massless tether with a masses tied to both ends rotates about a fixed axis through the center. Can the total acceleration of the tether/mass combination be zero if the angular velocity is constant?

### 10.3 Relating Angular and Translational Quantities

9. Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

10. In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

11. Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) the plate starts to spin faster? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

### 10.4 Moment of Inertia and Rotational Kinetic Energy

12. What if another planet the same size as Earth were put into orbit around the Sun along with Earth. Would the moment of inertia of the system increase, decrease, or stay the same?
13. A solid sphere is rotating about an axis through its center at a constant rotation rate. Another hollow sphere of the same mass and radius is rotating about its axis through the center at the same rotation rate. Which sphere has a greater rotational kinetic energy?

### 10.5 Calculating Moments of Inertia

14. If a child walks toward the center of a merry-go-round, does the moment of inertia increase or decrease?
15. A discus thrower rotates with a discus in his hand before letting it go. (a) How does his moment of inertia change after releasing the discus? (b) What would be a good approximation to use in calculating the moment of inertia of the discus thrower and discus?
16. Does increasing the number of blades on a propeller increase or decrease its moment of inertia, and why?
17. The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is  $mL^2/3$ . Why is this moment of inertia greater than it would be if you spun a point mass  $m$  at the location of the center of mass of the rod (at  $L/2$ ) (that would be  $mL^2/4$ )?

18. Why is the moment of inertia of a hoop that has a mass  $M$  and a radius  $R$  greater than the moment of inertia of a disk that has the same mass and radius?

### 10.6 Torque

19. What three factors affect the torque created by a force relative to a specific pivot point?

20. Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

21. When reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle's frame?

22. Can a single force produce a zero torque?

23. Can a set of forces have a net torque that is zero and a

net force that is not zero?

24. Can a set of forces have a net force that is zero and a net torque that is not zero?

25. In the expression  $\vec{r} \times \vec{F}$  can  $|\vec{r}|$  ever be less than the lever arm? Can it be equal to the lever arm?

### 10.7 Newton's Second Law for Rotation

26. If you were to stop a spinning wheel with a constant force, where on the wheel would you apply the force to produce the maximum negative acceleration?

27. A rod is pivoted about one end. Two forces  $\vec{F}$  and  $-\vec{F}$  are applied to it. Under what circumstances will the rod not rotate?

## PROBLEMS

### 10.1 Rotational Variables

28. Calculate the angular velocity of Earth.

29. A track star runs a 400-m race on a 400-m circular track in 45 s. What is his angular velocity assuming a constant speed?

30. A wheel rotates at a constant rate of  $2.0 \times 10^3$  rev/min. (a) What is its angular velocity in radians per second? (b) Through what angle does it turn in 10 s? Express the solution in radians and degrees.

31. A particle moves 3.0 m along a circle of radius 1.5 m. (a) Through what angle does it rotate? (b) If the particle makes this trip in 1.0 s at a constant speed, what is its angular velocity? (c) What is its acceleration?

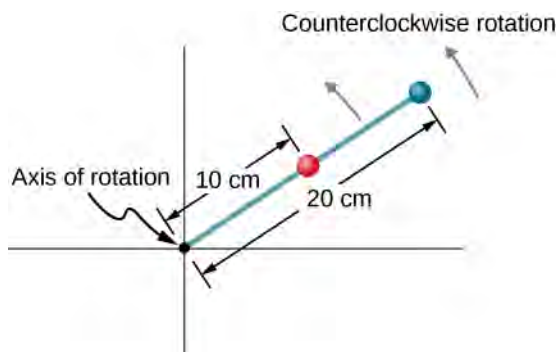
32. A compact disc rotates at 500 rev/min. If the diameter of the disc is 120 mm, (a) what is the tangential speed of a point at the edge of the disc? (b) At a point halfway to the center of the disc?

33. **Unreasonable results.** The propeller of an aircraft is spinning at 10 rev/s when the pilot shuts off the engine. The propeller reduces its angular velocity at a constant  $2.0 \text{ rad/s}^2$  for a time period of 40 s. What is the rotation rate of the propeller in 40 s? Is this a reasonable situation?

34. A gyroscope slows from an initial rate of 32.0 rad/s at a rate of  $0.700 \text{ rad/s}^2$ . How long does it take to come to rest?

35. On takeoff, the propellers on a UAV (unmanned aerial vehicle) increase their angular velocity from rest at a rate of  $\omega = (25.0t) \text{ rad/s}$  for 3.0 s. (a) What is the instantaneous angular velocity of the propellers at  $t = 2.0 \text{ s}$ ? (b) What is the angular acceleration?

36. The angular position of a rod varies as  $20.0t^2$  radians from time  $t = 0$ . The rod has two beads on it as shown in the following figure, one at 10 cm from the rotation axis and the other at 20 cm from the rotation axis. (a) What is the instantaneous angular velocity of the rod at  $t = 5 \text{ s}$ ? (b) What is the angular acceleration of the rod? (c) What are the tangential speeds of the beads at  $t = 5 \text{ s}$ ? (d) What are the tangential accelerations of the beads at  $t = 5 \text{ s}$ ? (e) What are the centripetal accelerations of the beads at  $t = 5 \text{ s}$ ?



## 10.2 Rotation with Constant Angular Acceleration

37. A wheel has a constant angular acceleration of  $5.0 \text{ rad/s}^2$ . Starting from rest, it turns through 300 rad. (a) What is its final angular velocity? (b) How much time elapses while it turns through the 300 radians?

38. During a 6.0-s time interval, a flywheel with a constant angular acceleration turns through 500 radians that acquire an angular velocity of 100 rad/s. (a) What is the angular velocity at the beginning of the 6.0 s? (b) What is the angular acceleration of the flywheel?

39. The angular velocity of a rotating rigid body increases from 500 to 1500 rev/min in 120 s. (a) What is the angular acceleration of the body? (b) Through what angle does it turn in this 120 s?

40. A flywheel slows from 600 to 400 rev/min while rotating through 40 revolutions. (a) What is the angular acceleration of the flywheel? (b) How much time elapses during the 40 revolutions?

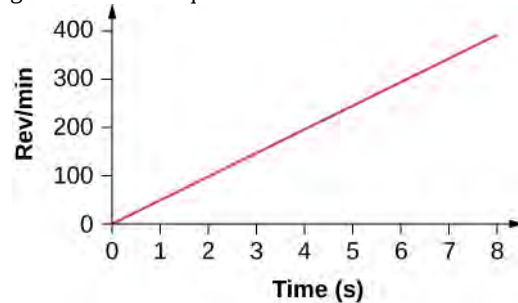
41. A wheel 1.0 m in diameter rotates with an angular acceleration of  $4.0 \text{ rad/s}^2$ . (a) If the wheel's initial angular velocity is  $2.0 \text{ rad/s}$ , what is its angular velocity after 10 s? (b) Through what angle does it rotate in the 10-s interval? (c) What are the tangential speed and acceleration of a point on the rim of the wheel at the end of the 10-s interval?

42. A vertical wheel with a diameter of 50 cm starts from rest and rotates with a constant angular acceleration of  $5.0 \text{ rad/s}^2$  around a fixed axis through its center counterclockwise. (a) Where is the point that is initially at the bottom of the wheel at  $t = 10 \text{ s}$ ? (b) What is the point's linear acceleration at this instant?

43. A circular disk of radius 10 cm has a constant angular acceleration of  $1.0 \text{ rad/s}^2$ ; at  $t = 0$  its angular velocity is  $2.0 \text{ rad/s}$ . (a) Determine the disk's angular velocity at  $t = 5.0 \text{ s}$ . (b) What is the angle it has rotated through

during this time? (c) What is the tangential acceleration of a point on the disk at  $t = 5.0 \text{ s}$ ?

44. The angular velocity vs. time for a fan on a hovercraft is shown below. (a) What is the angle through which the fan blades rotate in the first 8 seconds? (b) Verify your result using the kinematic equations.



45. A rod of length 20 cm has two beads attached to its ends. The rod with beads starts rotating from rest. If the beads are to have a tangential speed of 20 m/s in 7 s, what is the angular acceleration of the rod to achieve this?

## 10.3 Relating Angular and Translational Quantities

46. At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?

47. A man stands on a merry-go-round that is rotating at  $2.5 \text{ rad/s}$ . If the coefficient of static friction between the man's shoes and the merry-go-round is  $\mu_S = 0.5$ , how far from the axis of rotation can he stand without sliding?

48. An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min. (a) What is the average angular acceleration in  $\text{rad/s}^2$ ? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the centripetal acceleration in  $\text{m/s}^2$  and multiples of  $g$  of this point at full rpm? (d) What is the total distance travelled by a point 9.5 cm from the axis of rotation of the ultracentrifuge?

49. A wind turbine is rotating counterclockwise at  $0.5 \text{ rev/s}$  and slows to a stop in 10 s. Its blades are 20 m in length. (a) What is the angular acceleration of the turbine? (b) What is the centripetal acceleration of the tip of the blades at  $t = 0 \text{ s}$ ? (c) What is the magnitude and direction of the total linear acceleration of the tip of the blades at  $t = 0 \text{ s}$ ?

50. What is (a) the angular speed and (b) the linear speed of a point on Earth's surface at latitude  $30^\circ \text{ N}$ . Take the radius of the Earth to be 6309 km. (c) At what latitude

would your linear speed be 10 m/s?

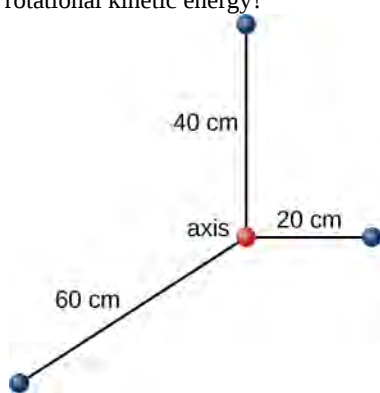
51. A child with mass 30 kg sits on the edge of a merry-go-round at a distance of 3.0 m from its axis of rotation. The merry-go-round accelerates from rest up to 0.4 rev/s in 10 s. If the coefficient of static friction between the child and the surface of the merry-go-round is 0.6, does the child fall off before 5 s?

52. A bicycle wheel with radius 0.3m rotates from rest to 3 rev/s in 5 s. What is the magnitude and direction of the total acceleration vector at the edge of the wheel at 1.0 s?

53. The angular velocity of a flywheel with radius 1.0 m varies according to  $\omega(t) = 2.0t$ . Plot  $a_c(t)$  and  $a_t(t)$  from  $t = 0$  to 3.0 s for  $r = 1.0$  m. Analyze these results to explain when  $a_c \gg a_t$  and when  $a_c \ll a_t$  for a point on the flywheel at a radius of 1.0 m.

### 10.4 Moment of Inertia and Rotational Kinetic Energy

54. A system of point particles is shown in the following figure. Each particle has mass 0.3 kg and they all lie in the same plane. (a) What is the moment of inertia of the system about the given axis? (b) If the system rotates at 5 rev/s, what is its rotational kinetic energy?



55. (a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

56. Calculate the rotational kinetic energy of a 12-kg motorcycle wheel if its angular velocity is 120 rad/s and its inner radius is 0.280 m and outer radius 0.330 m.

57. A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is  $0.500 \text{ kg}\cdot\text{m}^2$ , what is the rotational kinetic energy of the

forearm?

58. A diver goes into a somersault during a dive by tucking her limbs. If her rotational kinetic energy is 100 J and her moment of inertia in the tuck is  $9.0 \text{ kg}\cdot\text{m}^2$ , what is her rotational rate during the somersault?

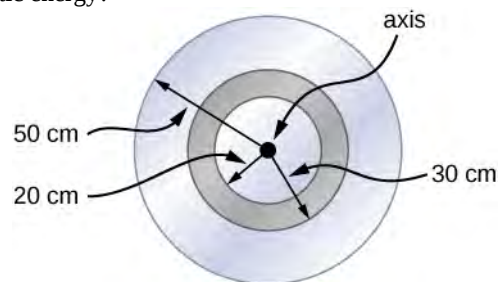
59. An aircraft is coming in for a landing at 300 meters height when the propeller falls off. The aircraft is flying at 40.0 m/s horizontally. The propeller has a rotation rate of 20 rev/s, a moment of inertia of  $70.0 \text{ kg}\cdot\text{m}^2$ , and a mass of 200 kg. Neglect air resistance. (a) With what translational velocity does the propeller hit the ground? (b) What is the rotation rate of the propeller at impact?

60. If air resistance is present in the preceding problem and reduces the propeller's rotational kinetic energy at impact by 30%, what is the propeller's rotation rate at impact?

61. A neutron star of mass  $2 \times 10^{30}$  kg and radius 10 km rotates with a period of 0.02 seconds. What is its rotational kinetic energy?

62. An electric sander consisting of a rotating disk of mass 0.7 kg and radius 10 cm rotates at 15 rev/sec. When applied to a rough wooden wall the rotation rate decreases by 20%. (a) What is the final rotational kinetic energy of the rotating disk? (b) How much has its rotational kinetic energy decreased?

63. A system consists of a disk of mass 2.0 kg and radius 50 cm upon which is mounted an annular cylinder of mass 1.0 kg with inner radius 20 cm and outer radius 30 cm (see below). The system rotates about an axis through the center of the disk and annular cylinder at 10 rev/s. (a) What is the moment of inertia of the system? (b) What is its rotational kinetic energy?



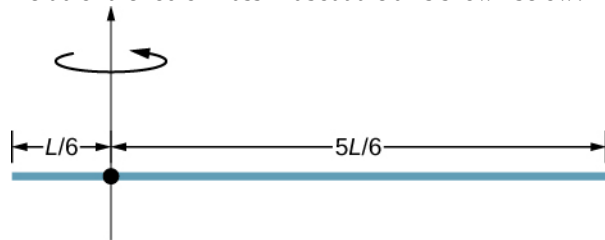
### 10.5 Calculating Moments of Inertia

64. While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is  $3.75 \text{ kg}\cdot\text{m}^2$  and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the



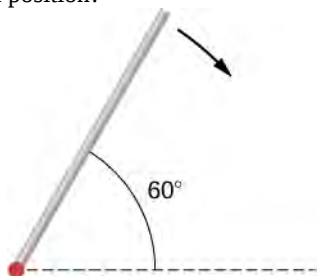
velocity of tip of the punter's shoe if it is 1.05 m from the hip joint?

65. Using the parallel axis theorem, what is the moment of inertia of the rod of mass  $m$  about the axis shown below?

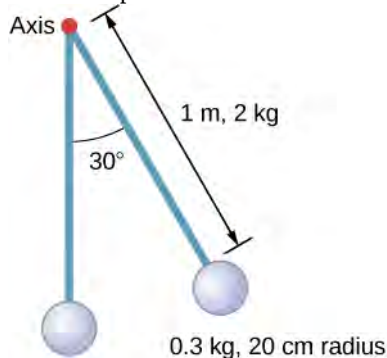


66. Find the moment of inertia of the rod in the previous problem by direct integration.

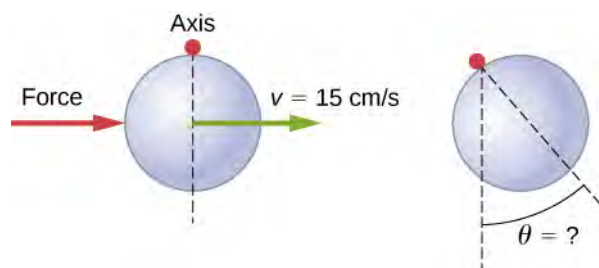
67. A uniform rod of mass 1.0 kg and length 2.0 m is free to rotate about one end (see the following figure). If the rod is released from rest at an angle of  $60^\circ$  with respect to the horizontal, what is the speed of the tip of the rod as it passes the horizontal position?



68. A pendulum consists of a rod of mass 2 kg and length 1 m with a solid sphere at one end with mass 0.3 kg and radius 20 cm (see the following figure). If the pendulum is released from rest at an angle of  $30^\circ$ , what is the angular velocity at the lowest point?



69. A solid sphere of radius 10 cm is allowed to rotate freely about an axis. The sphere is given a sharp blow so that its center of mass starts from the position shown in the following figure with speed 15 cm/s. What is the maximum angle that the diameter makes with the vertical?

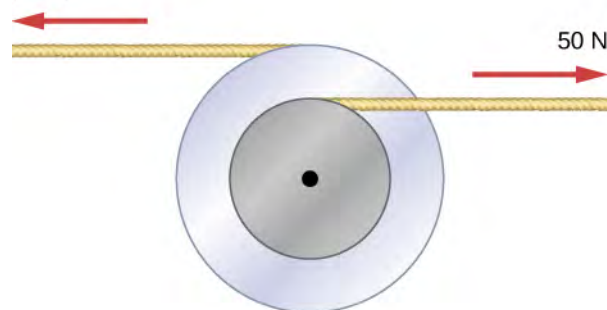


70. Calculate the moment of inertia by direct integration of a thin rod of mass  $M$  and length  $L$  about an axis through the rod at  $L/3$ , as shown below. Check your answer with the parallel-axis theorem.



## 10.6 Torque

71. Two flywheels of negligible mass and different radii are bonded together and rotate about a common axis (see below). The smaller flywheel of radius 30 cm has a cord that has a pulling force of 50 N on it. What pulling force needs to be applied to the cord connecting the larger flywheel of radius 50 cm such that the combination does not rotate?  
 $F = ?$



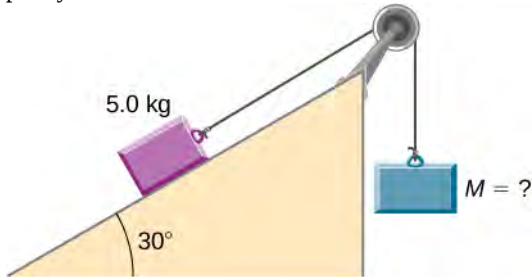
72. The cylindrical head bolts on a car are to be tightened with a torque of  $62.0 \text{ N} \cdot \text{m}$ . If a mechanic uses a wrench of length 20 cm, what perpendicular force must he exert on the end of the wrench to tighten a bolt correctly?

73. (a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850 m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges? There is only one pair of hinges.

74. When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. How much torque are you exerting in newton-meters (relative to the center of the bolt)?

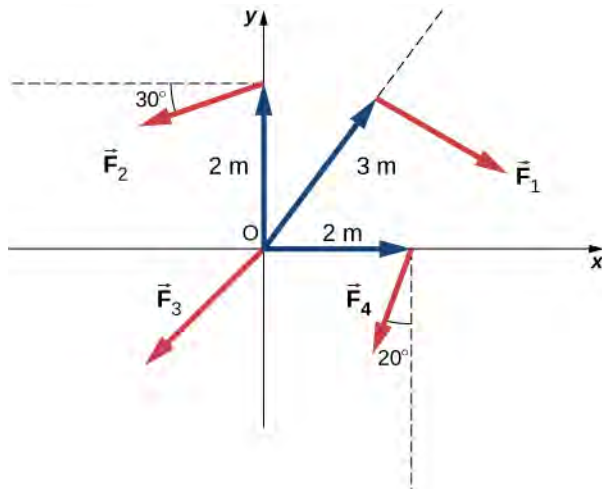
75. What hanging mass must be placed on the cord to keep the pulley from rotating (see the following figure)? The

mass on the frictionless plane is 5.0 kg. The inner radius of the pulley is 20 cm and the outer radius is 30 cm.

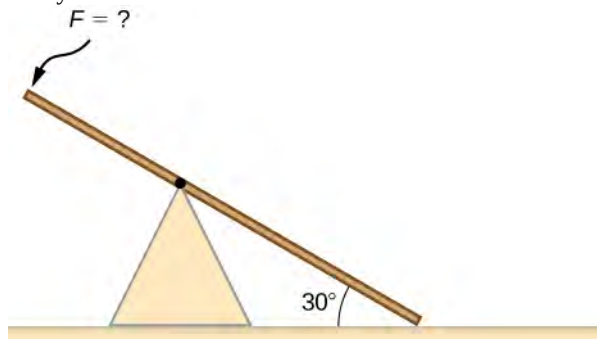


76. A simple pendulum consists of a massless tether 50 cm in length connected to a pivot and a small mass of 1.0 kg attached at the other end. What is the torque about the pivot when the pendulum makes an angle of  $40^\circ$  with respect to the vertical?

77. Calculate the torque about the z-axis that is out of the page at the origin in the following figure, given that  $F_1 = 3 \text{ N}$ ,  $F_2 = 2 \text{ N}$ ,  $F_3 = 3 \text{ N}$ ,  $F_4 = 1.8 \text{ N}$ .



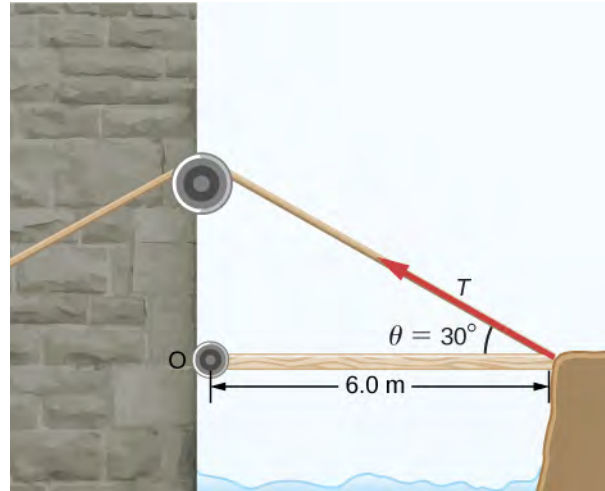
78. A seesaw has length 10.0 m and uniform mass 10.0 kg and is resting at an angle of  $30^\circ$  with respect to the ground (see the following figure). The pivot is located at 6.0 m. What magnitude of force needs to be applied perpendicular to the seesaw at the raised end so as to allow the seesaw to barely start to rotate?



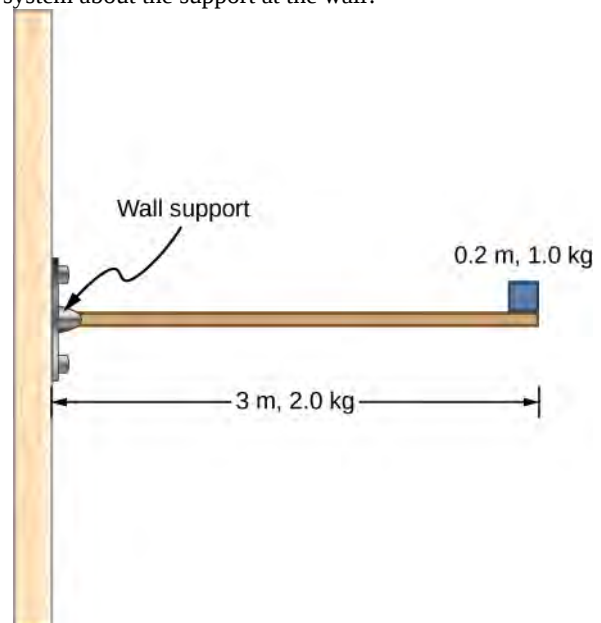
79. A pendulum consists of a rod of mass 1 kg and length

1 m connected to a pivot with a solid sphere attached at the other end with mass 0.5 kg and radius 30 cm. What is the torque about the pivot when the pendulum makes an angle of  $30^\circ$  with respect to the vertical?

80. A torque of  $5.00 \times 10^3 \text{ N} \cdot \text{m}$  is required to raise a drawbridge (see the following figure). What is the tension necessary to produce this torque? Would it be easier to raise the drawbridge if the angle  $\theta$  were larger or smaller?



81. A horizontal beam of length 3 m and mass 2.0 kg has a mass of 1.0 kg and width 0.2 m sitting at the end of the beam (see the following figure). What is the torque of the system about the support at the wall?



82. What force must be applied to end of a rod along the x-axis of length 2.0 m in order to produce a torque on the rod about the origin of  $8.0 \hat{k} \text{ N} \cdot \text{m}$ ?

83. What is the torque about the origin of the force



$(5.0\hat{i} - 2.0\hat{j} + 1.0\hat{k})\text{ N}$  if it is applied at the point whose position is:  $\vec{r} = (-2.0\hat{i} + 4.0\hat{j})\text{ m}$ ?

### 10.7 Newton's Second Law for Rotation

**84.** You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

**85.** Suppose you exert a force of 180 N tangential to a 0.280-m-radius, 75.0-kg grindstone (a solid disk). (a) What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?

**86.** A flywheel ( $I = 50\text{ kg}\cdot\text{m}^2$ ) starting from rest acquires an angular velocity of 200.0 rad/s while subject to a constant torque from a motor for 5 s. (a) What is the angular acceleration of the flywheel? (b) What is the magnitude of the torque?

**87.** A constant torque is applied to a rigid body whose moment of inertia is  $4.0\text{ kg}\cdot\text{m}^2$  around the axis of rotation. If the wheel starts from rest and attains an angular velocity of 20.0 rad/s in 10.0 s, what is the applied torque?

**88.** A torque of 50.0 N·m is applied to a grinding wheel ( $I = 20.0\text{ kg}\cdot\text{m}^2$ ) for 20 s. (a) If it starts from rest, what is the angular velocity of the grinding wheel after the torque is removed? (b) Through what angle does the wheel move through while the torque is applied?

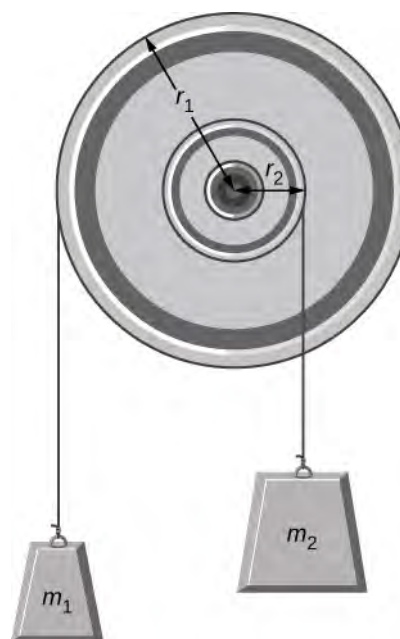
**89.** A flywheel ( $I = 100.0\text{ kg}\cdot\text{m}^2$ ) rotating at 500.0 rev/min is brought to rest by friction in 2.0 min. What is the frictional torque on the flywheel?

**90.** A uniform cylindrical grinding wheel of mass 50.0 kg and diameter 1.0 m is turned on by an electric motor. The friction in the bearings is negligible. (a) What torque must be applied to the wheel to bring it from rest to 120 rev/min in 20 revolutions? (b) A tool whose coefficient of kinetic friction with the wheel is 0.60 is pressed perpendicularly against the wheel with a force of 40.0 N. What torque must be supplied by the motor to keep the wheel rotating at a constant angular velocity?

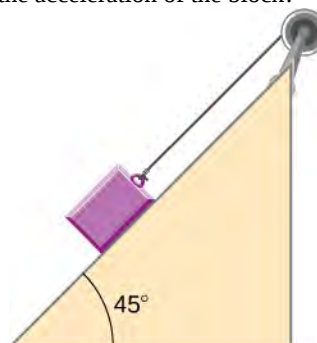
**91.** Suppose when Earth was created, it was not rotating.

However, after the application of a uniform torque after 6 days, it was rotating at 1 rev/day. (a) What was the angular acceleration during the 6 days? (b) What torque was applied to Earth during this period? (c) What force tangent to Earth at its equator would produce this torque?

**92.** A pulley of moment of inertia  $2.0\text{ kg}\cdot\text{m}^2$  is mounted on a wall as shown in the following figure. Light strings are wrapped around two circumferences of the pulley and weights are attached. What are (a) the angular acceleration of the pulley and (b) the linear acceleration of the weights? Assume the following data:  $r_1 = 50\text{ cm}$ ,  $r_2 = 20\text{ cm}$ ,  $m_1 = 1.0\text{ kg}$ ,  $m_2 = 2.0\text{ kg}$

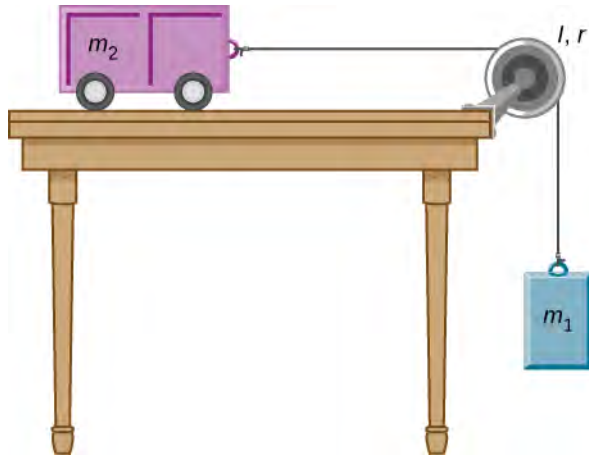


**93.** A block of mass 3 kg slides down an inclined plane at an angle of  $45^\circ$  with a massless tether attached to a pulley with mass 1 kg and radius 0.5 m at the top of the incline (see the following figure). The pulley can be approximated as a disk. The coefficient of kinetic friction on the plane is 0.4. What is the acceleration of the block?



**94.** The cart shown below moves across the table top as the block falls. What is the acceleration of the cart? Neglect friction and assume the following data:

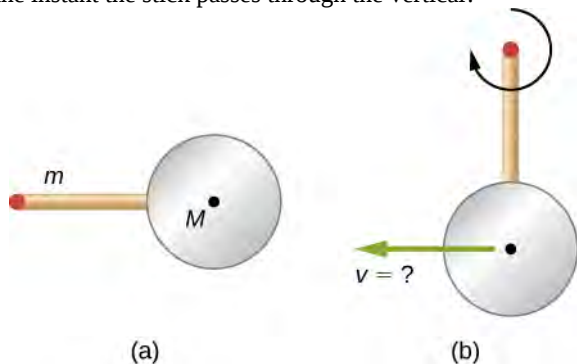
$$m_1 = 2.0 \text{ kg}, m_2 = 4.0 \text{ kg}, I = 0.4 \text{ kg}\cdot\text{m}^2, r = 20 \text{ cm}$$



95. A uniform rod of mass and length is held vertically by two strings of negligible mass, as shown below. (a) Immediately after the string is cut, what is the linear acceleration of the free end of the stick? (b) Of the middle of the stick?



96. A thin stick of mass  $0.2 \text{ kg}$  and length  $L = 0.5 \text{ m}$  is attached to the rim of a metal disk of mass  $M = 2.0 \text{ kg}$  and radius  $R = 0.3 \text{ m}$ . The stick is free to rotate around a horizontal axis through its other end (see the following figure). (a) If the combination is released with the stick horizontal, what is the speed of the center of the disk when the stick is vertical? (b) What is the acceleration of the center of the disk at the instant the stick is released? (c) At the instant the stick passes through the vertical?



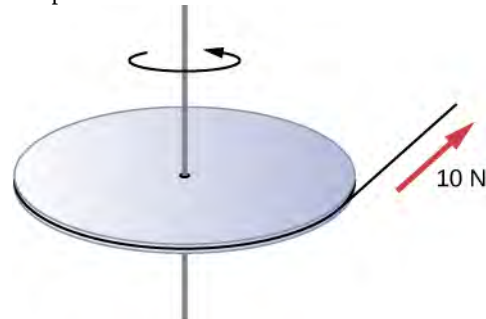
### 10.8 Work and Power for Rotational Motion

97. A wind turbine rotates at  $20 \text{ rev/min}$ . If its power output is  $2.0 \text{ MW}$ , what is the torque produced on the turbine from the wind?

98. A clay cylinder of radius  $20 \text{ cm}$  on a potter's wheel spins at a constant rate of  $10 \text{ rev/s}$ . The potter applies a force of  $10 \text{ N}$  to the clay with his hands where the coefficient of friction is  $0.1$  between his hands and the clay. What is the power that the potter has to deliver to the wheel to keep it rotating at this constant rate?

99. A uniform cylindrical grindstone has a mass of  $10 \text{ kg}$  and a radius of  $12 \text{ cm}$ . (a) What is the rotational kinetic energy of the grindstone when it is rotating at  $1.5 \times 10^3 \text{ rev/min}$ ? (b) After the grindstone's motor is turned off, a knife blade is pressed against the outer edge of the grindstone with a perpendicular force of  $5.0 \text{ N}$ . The coefficient of kinetic friction between the grindstone and the blade is  $0.80$ . Use the work energy theorem to determine how many turns the grindstone makes before it stops.

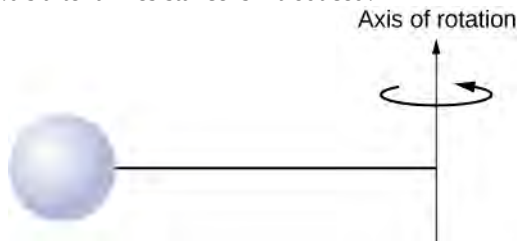
100. A uniform disk of mass  $500 \text{ kg}$  and radius  $0.25 \text{ m}$  is mounted on frictionless bearings so it can rotate freely around a vertical axis through its center (see the following figure). A cord is wrapped around the rim of the disk and pulled with a force of  $10 \text{ N}$ . (a) How much work has the force done at the instant the disk has completed three revolutions, starting from rest? (b) Determine the torque due to the force, then calculate the work done by this torque at the instant the disk has completed three revolutions? (c) What is the angular velocity at that instant? (d) What is the power output of the force at that instant?



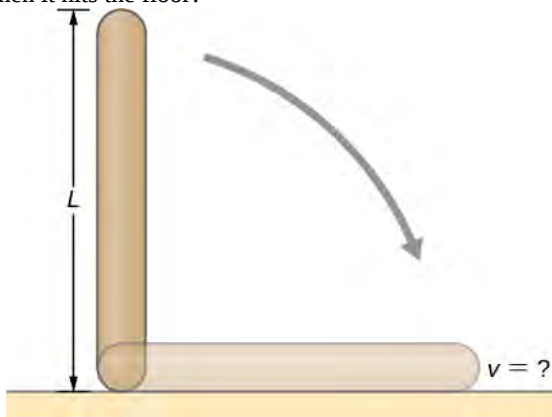
101. A propeller is accelerated from rest to an angular velocity of  $1000 \text{ rev/min}$  over a period of  $6.0 \text{ seconds}$  by a constant torque of  $2.0 \times 10^3 \text{ N}\cdot\text{m}$ . (a) What is the moment of inertia of the propeller? (b) What power is being provided to the propeller  $3.0 \text{ s}$  after it starts rotating?

102. A sphere of mass  $1.0 \text{ kg}$  and radius  $0.5 \text{ m}$  is attached to the end of a massless rod of length  $3.0 \text{ m}$ . The rod rotates about an axis that is at the opposite end of the sphere (see below). The system rotates horizontally about the axis at a

constant 400 rev/min. After rotating at this angular speed in a vacuum, air resistance is introduced and provides a force 0.15 N on the sphere opposite to the direction of motion. What is the power provided by air resistance to the system 100.0 s after air resistance is introduced?



**103.** A uniform rod of length  $L$  and mass  $M$  is held vertically with one end resting on the floor as shown below. When the rod is released, it rotates around its lower end until it hits the floor. Assuming the lower end of the rod does not slip, what is the linear velocity of the upper end when it hits the floor?



**104.** An athlete in a gym applies a constant force of 50 N to the pedals of a bicycle to keep the rotation rate of the wheel at 10 rev/s. The length of the pedal arms is 30 cm. What is the power delivered to the bicycle by the athlete?

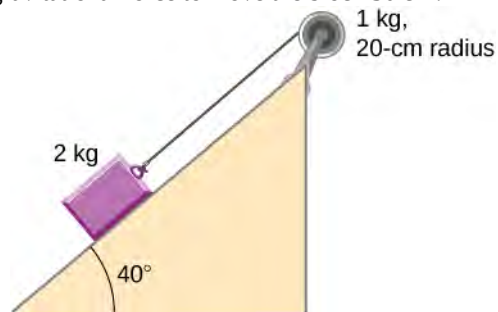
## ADDITIONAL PROBLEMS

**107.** A cyclist is riding such that the wheels of the bicycle have a rotation rate of 3.0 rev/s. If the cyclist brakes such that the rotation rate of the wheels decrease at a rate of  $0.3 \text{ rev/s}^2$ , how long does it take for the cyclist to come to a complete stop?

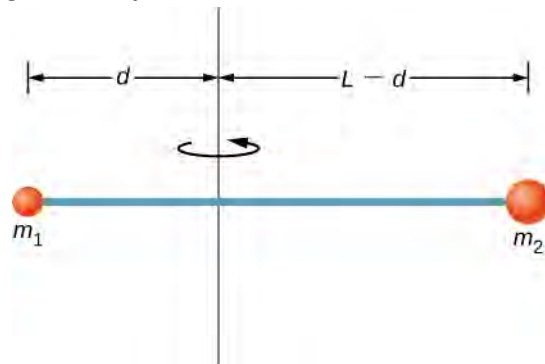
**108.** Calculate the angular velocity of the orbital motion of Earth around the Sun.

**109.** A phonograph turntable rotating at  $33 \frac{1}{3}$  rev/min slows down and stops in 1.0 min. (a) What is the turntable's angular acceleration assuming it is constant? (b) How many revolutions does the turntable make while stopping?

**105.** A 2-kg block on a frictionless inclined plane at  $40^\circ$  has a cord attached to a pulley of mass 1 kg and radius 20 cm (see the following figure). (a) What is the acceleration of the block down the plane? (b) What is the work done by the gravitational force to move the block 50 cm?



**106.** Small bodies of mass  $m_1$  and  $m_2$  are attached to opposite ends of a thin rigid rod of length  $L$  and mass  $M$ . The rod is mounted so that it is free to rotate in a horizontal plane around a vertical axis (see below). What distance  $d$  from  $m_1$  should the rotational axis be so that a minimum amount of work is required to set the rod rotating at an angular velocity  $\omega$ ?



**110.** With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s under a constant angular acceleration. (a) What is its angular acceleration in  $\text{rad/s}^2$ ? (b) How many revolutions does it go through in the process?

**111.** Suppose a piece of dust has fallen on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

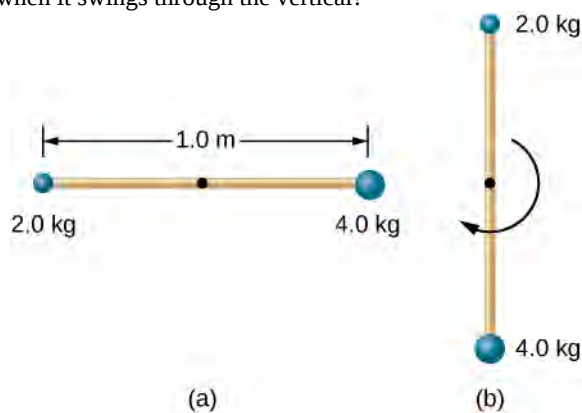
**112.** A system of point particles is rotating about a fixed axis at 4 rev/s. The particles are fixed with respect to each other. The masses and distances to the axis of the point

particles are  $m_1 = 0.1 \text{ kg}$ ,  $r_1 = 0.2 \text{ m}$ ,  $m_2 = 0.05 \text{ kg}$ ,  $r_2 = 0.4 \text{ m}$ ,  $m_3 = 0.5 \text{ kg}$ ,  $r_3 = 0.01 \text{ m}$ .

(a) What is the moment of inertia of the system? (b) What is the rotational kinetic energy of the system?

**113.** Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximated by a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

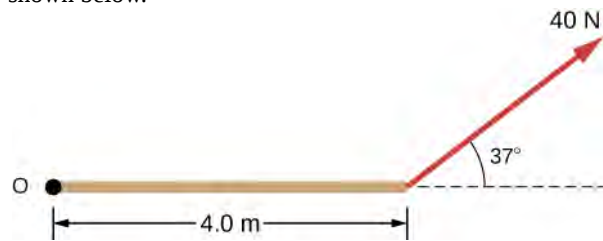
**114.** A stick of length 1.0 m and mass 6.0 kg is free to rotate about a horizontal axis through the center. Small bodies of masses 4.0 and 2.0 kg are attached to its two ends (see the following figure). The stick is released from the horizontal position. What is the angular velocity of the stick when it swings through the vertical?



**115.** A pendulum consists of a rod of length 2 m and mass 3 kg with a solid sphere of mass 1 kg and radius 0.3 m attached at one end. The axis of rotation is as shown below. What is the angular velocity of the pendulum at its lowest point if it is released from rest at an angle of  $30^\circ$ ?



**116.** Calculate the torque of the 40-N force around the axis through  $O$  and perpendicular to the plane of the page as shown below.



**117.** Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.

**118.** The force of  $20 \hat{j} \text{ N}$  is applied at  $\vec{r} = (4.0 \hat{i} - 2.0 \hat{j}) \text{ m}$ . What is the torque of this force about the origin?

**119.** An automobile engine can produce  $200 \text{ N} \cdot \text{m}$  of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0-kg disk that has a 0.180-m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.

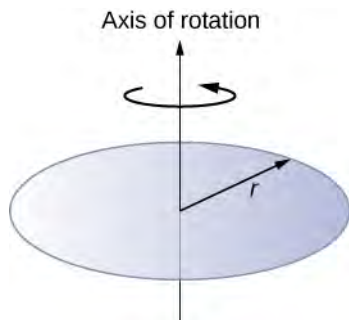
**120.** A grindstone with a mass of 50 kg and radius 0.8 m maintains a constant rotation rate of 4.0 rev/s by a motor while a knife is pressed against the edge with a force of 5.0 N. The coefficient of kinetic friction between the grindstone and the blade is 0.8. What is the power provided by the motor to keep the grindstone at the constant rotation rate?

## CHALLENGE PROBLEMS

**121.** The angular acceleration of a rotating rigid body is given by  $\alpha = (2.0 - 3.0t) \text{ rad/s}^2$ . If the body starts rotating from rest at  $t = 0$ , (a) what is the angular velocity? (b) Angular position? (c) What angle does it rotate through in 10 s? (d) Where does the vector perpendicular to the axis of rotation indicating  $0^\circ$  at  $t = 0$  lie at  $t = 10 \text{ s}$ ?

**122.** Earth's day has increased by 0.002 s in the last century. If this increase in Earth's period is constant, how long will it take for Earth to come to rest?

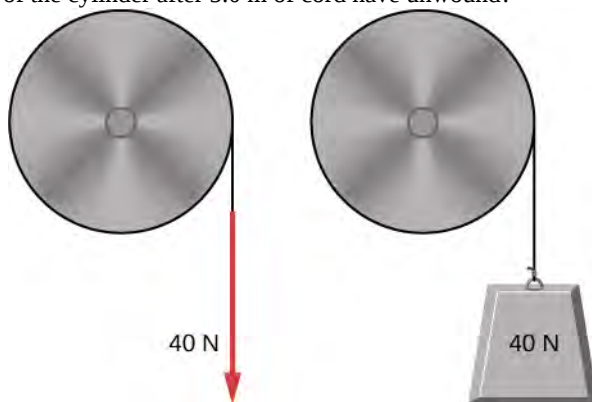
**123.** A disk of mass  $m$ , radius  $R$ , and area  $A$  has a surface mass density  $\sigma = \frac{mR}{AR}$  (see the following figure). What is the moment of inertia of the disk about an axis through the center?



**124.** Zorch, an archenemy of Rotation Man, decides to slow Earth's rotation to once per 28.0 h by exerting an

opposing force at and parallel to the equator. Rotation Man is not immediately concerned, because he knows Zorch can only exert a force of  $4.00 \times 10^7 \text{ N}$  (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Rotation Man time to devote to other villains.)

**125.** A cord is wrapped around the rim of a solid cylinder of radius 0.25 m, and a constant force of 40 N is exerted on the cord shown, as shown in the following figure. The cylinder is mounted on frictionless bearings, and its moment of inertia is  $6.0 \text{ kg} \cdot \text{m}^2$ . (a) Use the work energy theorem to calculate the angular velocity of the cylinder after 5.0 m of cord have been removed. (b) If the 40-N force is replaced by a 40-N weight, what is the angular velocity of the cylinder after 5.0 m of cord have unwound?







# 11 | ANGULAR MOMENTUM



**Figure 11.1** A helicopter has its main lift blades rotating to keep the aircraft airborne. Due to conservation of angular momentum, the body of the helicopter would want to rotate in the opposite sense to the blades, if it were not for the small rotor on the tail of the aircraft, which provides thrust to stabilize it.

## Chapter Outline

- 11.1 Rolling Motion
- 11.2 Angular Momentum
- 11.3 Conservation of Angular Momentum
- 11.4 Precession of a Gyroscope

## Introduction

Angular momentum is the rotational counterpart of linear momentum. Any massive object that rotates about an axis carries angular momentum, including rotating flywheels, planets, stars, hurricanes, tornadoes, whirlpools, and so on. The helicopter shown in the chapter-opening picture can be used to illustrate the concept of angular momentum. The lift blades spin about a vertical axis through the main body and carry angular momentum. The body of the helicopter tends to rotate in the opposite sense in order to conserve angular momentum. The small rotors at the tail of the aircraft provide a counter thrust against the body to prevent this from happening, and the helicopter stabilizes itself. The concept of conservation of angular momentum is discussed later in this chapter. In the main part of this chapter, we explore the intricacies of angular momentum of rigid bodies such as a top, and also of point particles and systems of particles. But to be complete, we start with a discussion of rolling motion, which builds upon the concepts of the previous chapter.

## 11.1 | Rolling Motion

### Learning Objectives

By the end of this section, you will be able to:

- Describe the physics of rolling motion without slipping
- Explain how linear variables are related to angular variables for the case of rolling motion without slipping
- Find the linear and angular accelerations in rolling motion with and without slipping
- Calculate the static friction force associated with rolling motion without slipping
- Use energy conservation to analyze rolling motion

Rolling motion is that common combination of rotational and translational motion that we see everywhere, every day. Think about the different situations of wheels moving on a car along a highway, or wheels on a plane landing on a runway, or wheels on a robotic explorer on another planet. Understanding the forces and torques involved in **rolling motion** is a crucial factor in many different types of situations.

For analyzing rolling motion in this chapter, refer to **Figure 10.20** in **Fixed-Axis Rotation** to find moments of inertia of some common geometrical objects. You may also find it useful in other calculations involving rotation.

### Rolling Motion without Slipping

People have observed rolling motion without slipping ever since the invention of the wheel. For example, we can look at the interaction of a car's tires and the surface of the road. If the driver depresses the accelerator to the floor, such that the tires spin without the car moving forward, there must be kinetic friction between the wheels and the surface of the road. If the driver depresses the accelerator slowly, causing the car to move forward, then the tires roll without slipping. It is surprising to most people that, in fact, the bottom of the wheel is at rest with respect to the ground, indicating there must be static friction between the tires and the road surface. In **Figure 11.2**, the bicycle is in motion with the rider staying upright. The tires have contact with the road surface, and, even though they are rolling, the bottoms of the tires deform slightly, do not slip, and are at rest with respect to the road surface for a measurable amount of time. There must be static friction between the tire and the road surface for this to be so.



(a)

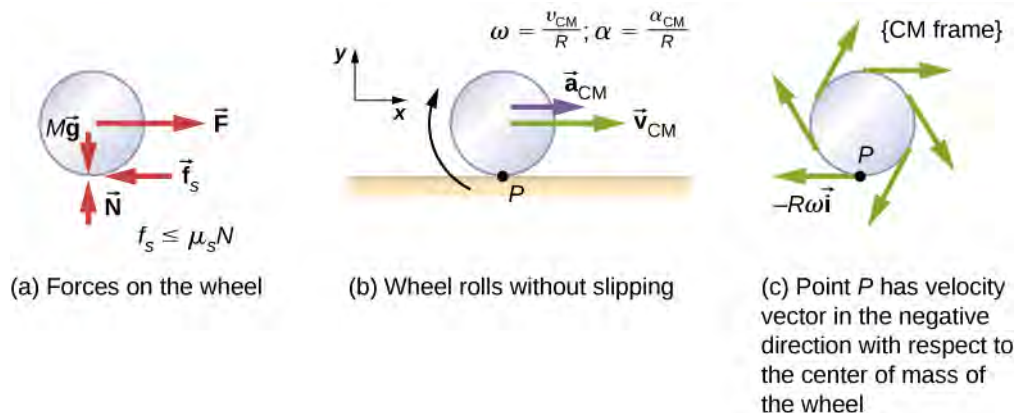


(b)

**Figure 11.2** (a) The bicycle moves forward, and its tires do not slip. The bottom of the slightly deformed tire is at rest with respect to the road surface for a measurable amount of time. (b) This image shows that the top of a rolling wheel appears blurred by its motion, but the bottom of the wheel is instantaneously at rest. (credit a: modification of work by Nelson Lourenço; credit b: modification of work by Colin Rose)



To analyze rolling without slipping, we first derive the linear variables of velocity and acceleration of the center of mass of the wheel in terms of the angular variables that describe the wheel's motion. The situation is shown in **Figure 11.3**.



**Figure 11.3** (a) A wheel is pulled across a horizontal surface by a force  $\vec{F}$ . The force of static friction  $\vec{f}_s$ ,  $|\vec{f}_s| \leq \mu_s N$  is large enough to keep it from slipping. (b) The linear velocity and acceleration vectors of the center of mass and the relevant expressions for  $\omega$  and  $\alpha$ . Point  $P$  is at rest relative to the surface. (c) Relative to the center of mass (CM) frame, point  $P$  has linear velocity  $-R\omega \hat{i}$ .

From **Figure 11.3(a)**, we see the force vectors involved in preventing the wheel from slipping. In (b), point  $P$  that touches the surface is at rest relative to the surface. Relative to the center of mass, point  $P$  has velocity  $-R\omega \hat{i}$ , where  $R$  is the radius of the wheel and  $\omega$  is the wheel's angular velocity about its axis. Since the wheel is rolling, the velocity of  $P$  with respect to the surface is its velocity with respect to the center of mass plus the velocity of the center of mass with respect to the surface:

$$\vec{v}_P = -R\omega \hat{i} + v_{CM} \hat{i}.$$

Since the velocity of  $P$  relative to the surface is zero,  $v_P = 0$ , this says that

$$v_{CM} = R\omega. \tag{11.1}$$

Thus, the velocity of the wheel's center of mass is its radius times the angular velocity about its axis. We show the correspondence of the linear variable on the left side of the equation with the angular variable on the right side of the equation. This is done below for the linear acceleration.

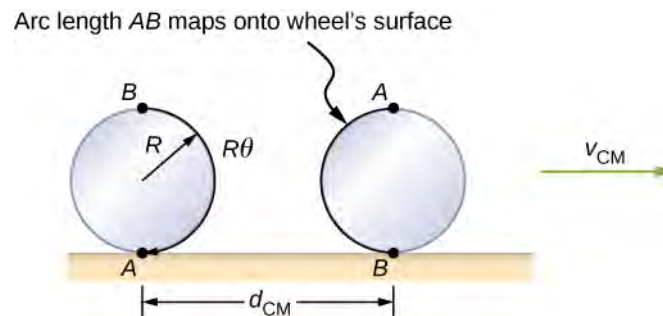
If we differentiate **Equation 11.1** on the left side of the equation, we obtain an expression for the linear acceleration of the center of mass. On the right side of the equation,  $R$  is a constant and since  $\alpha = \frac{d\omega}{dt}$ , we have

$$a_{CM} = R\alpha. \tag{11.2}$$

Furthermore, we can find the distance the wheel travels in terms of angular variables by referring to **Figure 11.4**. As the wheel rolls from point  $A$  to point  $B$ , its outer surface maps onto the ground by exactly the distance travelled, which is  $d_{CM}$ . We see from **Figure 11.4** that the length of the outer surface that maps onto the ground is the arc length  $R\theta$ . Equating the

two distances, we obtain

$$d_{\text{CM}} = R\theta. \quad (11.3)$$



**Figure 11.4** As the wheel rolls on the surface, the arc length  $R\theta$  from  $A$  to  $B$  maps onto the surface, corresponding to the distance  $d_{\text{CM}}$  that the center of mass has moved.

## Example 11.1

### Rolling Down an Inclined Plane

A solid cylinder rolls down an inclined plane without slipping, starting from rest. It has mass  $m$  and radius  $r$ . (a) What is its acceleration? (b) What condition must the coefficient of static friction  $\mu_S$  satisfy so the cylinder does not slip?

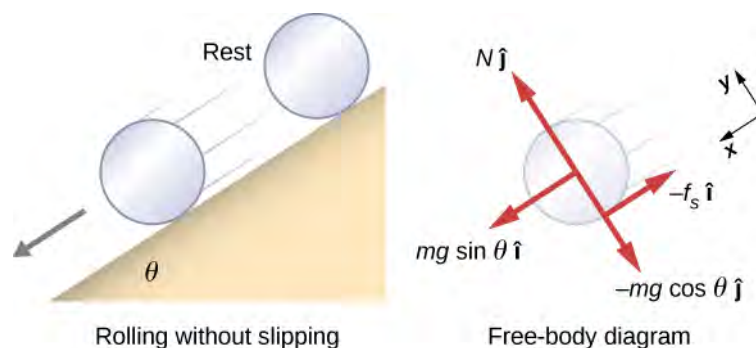
#### Strategy

Draw a sketch and free-body diagram, and choose a coordinate system. We put  $x$  in the direction down the plane and  $y$  upward perpendicular to the plane. Identify the forces involved. These are the normal force, the force of gravity, and the force due to friction. Write down Newton's laws in the  $x$ - and  $y$ -directions, and Newton's law for rotation, and then solve for the acceleration and force due to friction.

#### Solution

- a. The free-body diagram and sketch are shown in **Figure 11.5**, including the normal force, components of the weight, and the static friction force. There is barely enough friction to keep the cylinder rolling without slipping. Since there is no slipping, the magnitude of the friction force is less than or equal to  $\mu_S N$ . Writing down Newton's laws in the  $x$ - and  $y$ -directions, we have

$$\sum F_x = ma_x; \quad \sum F_y = ma_y.$$



**Figure 11.5** A solid cylinder rolls down an inclined plane without slipping from rest. The coordinate system has  $x$  in the direction down the inclined plane and  $y$  perpendicular to the plane. The free-body diagram is shown with the normal force, the static friction force, and the components of the weight  $m \vec{g}$ . Friction makes the cylinder roll down the plane rather than slip.

Substituting in from the free-body diagram,

$$\begin{aligned} mg \sin \theta - f_S &= m(a_{\text{CM}})_x, \\ N - mg \cos \theta &= 0, \\ f_S &\leq \mu_S N, \end{aligned}$$

we can then solve for the linear acceleration of the center of mass from these equations:

$$(a_{\text{CM}})_x = g(\sin \theta - \mu_S \cos \theta).$$

However, it is useful to express the linear acceleration in terms of the moment of inertia. For this, we write down Newton's second law for rotation,

$$\sum \tau_{\text{CM}} = I_{\text{CM}} \alpha.$$

The torques are calculated about the axis through the center of mass of the cylinder. The only nonzero torque is provided by the friction force. We have

$$f_S r = I_{\text{CM}} \alpha.$$

Finally, the linear acceleration is related to the angular acceleration by

$$(a_{\text{CM}})_x = r\alpha.$$

These equations can be used to solve for  $a_{\text{CM}}$ ,  $\alpha$ , and  $f_S$  in terms of the moment of inertia, where we have dropped the  $x$ -subscript. We write  $a_{\text{CM}}$  in terms of the vertical component of gravity and the friction force, and make the following substitutions.

$$a_{\text{CM}} = g \sin \theta - \frac{f_S}{m}$$

$$f_S = \frac{I_{\text{CM}} \alpha}{r} = \frac{I_{\text{CM}} a_{\text{CM}}}{r^2}$$

From this we obtain

$$\begin{aligned} a_{\text{CM}} &= g \sin \theta - \frac{I_{\text{CM}} a_{\text{CM}}}{mr^2}, \\ &= \frac{mg \sin \theta}{m + (I_{\text{CM}}/r^2)}. \end{aligned}$$

Note that this result is independent of the coefficient of static friction,  $\mu_S$ .

Since we have a solid cylinder, from **Figure 10.20**, we have  $I_{\text{CM}} = mr^2/2$  and

$$a_{\text{CM}} = \frac{mg \sin \theta}{m + (mr^2/2r^2)} = \frac{2}{3}g \sin \theta.$$

Therefore, we have

$$\alpha = \frac{a_{\text{CM}}}{r} = \frac{2}{3r}g \sin \theta.$$

- b. Because slipping does not occur,  $f_S \leq \mu_S N$ . Solving for the friction force,

$$f_S = I_{\text{CM}} \frac{\alpha}{r} = I_{\text{CM}} \frac{(a_{\text{CM}})}{r^2} = \frac{I_{\text{CM}}}{r^2} \left( \frac{mg \sin \theta}{m + (I_{\text{CM}}/r^2)} \right) = \frac{mg I_{\text{CM}} \sin \theta}{mr^2 + I_{\text{CM}}}.$$

Substituting this expression into the condition for no slipping, and noting that  $N = mg \cos \theta$ , we have

$$\frac{mg I_{\text{CM}} \sin \theta}{mr^2 + I_{\text{CM}}} \leq \mu_S mg \cos \theta$$

or

$$\mu_S \geq \frac{\tan \theta}{1 + (mr^2/I_{\text{CM}})}.$$

For the solid cylinder, this becomes

$$\mu_S \geq \frac{\tan \theta}{1 + (2mr^2/mr^2)} = \frac{1}{3} \tan \theta.$$

### Significance

- The linear acceleration is linearly proportional to  $\sin \theta$ . Thus, the greater the angle of the incline, the greater the linear acceleration, as would be expected. The angular acceleration, however, is linearly proportional to  $\sin \theta$  and inversely proportional to the radius of the cylinder. Thus, the larger the radius, the smaller the angular acceleration.
- For no slipping to occur, the coefficient of static friction must be greater than or equal to  $(1/3)\tan \theta$ . Thus, the greater the angle of incline, the greater the coefficient of static friction must be to prevent the cylinder from slipping.



**11.1 Check Your Understanding** A hollow cylinder is on an incline at an angle of  $60^\circ$ . The coefficient of static friction on the surface is  $\mu_S = 0.6$ . (a) Does the cylinder roll without slipping? (b) Will a solid cylinder roll without slipping?

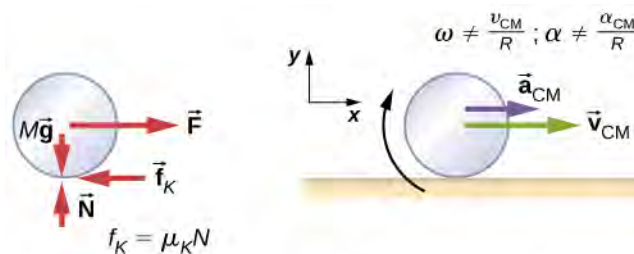
It is worthwhile to repeat the equation derived in this example for the acceleration of an object rolling without slipping:

$$a_{\text{CM}} = \frac{mg \sin \theta}{m + (I_{\text{CM}}/r^2)}. \quad (11.4)$$

This is a very useful equation for solving problems involving rolling without slipping. Note that the acceleration is less than that for an object sliding down a frictionless plane with no rotation. The acceleration will also be different for two rotating cylinders with different rotational inertias.

## Rolling Motion with Slipping

In the case of rolling motion with slipping, we must use the coefficient of kinetic friction, which gives rise to the kinetic friction force since static friction is not present. The situation is shown in **Figure 11.6**. In the case of slipping,  $v_{\text{CM}} - R\omega \neq 0$ , because point  $P$  on the wheel is not at rest on the surface, and  $v_P \neq 0$ . Thus,  $\omega \neq \frac{v_{\text{CM}}}{R}$ ,  $\alpha \neq \frac{a_{\text{CM}}}{R}$ .



(a) Forces on wheel

(b) Wheel is rolling and slipping

**Figure 11.6** (a) Kinetic friction arises between the wheel and the surface because the wheel is slipping. (b) The simple relationships between the linear and angular variables are no longer valid.

## Example 11.2

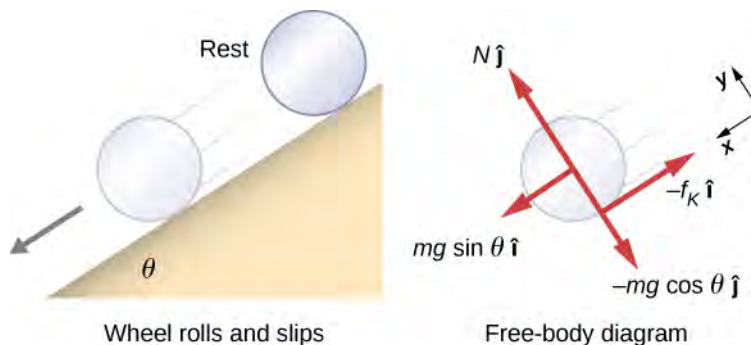
### Rolling Down an Inclined Plane with Slipping

A solid cylinder rolls down an inclined plane from rest and undergoes slipping (**Figure 11.7**). It has mass  $m$  and radius  $r$ . (a) What is its linear acceleration? (b) What is its angular acceleration about an axis through the center of mass?

#### Strategy

Draw a sketch and free-body diagram showing the forces involved. The free-body diagram is similar to the no-slipping case except for the friction force, which is kinetic instead of static. Use Newton's second law to solve for the acceleration in the  $x$ -direction. Use Newton's second law of rotation to solve for the angular acceleration.

## Solution



**Figure 11.7** A solid cylinder rolls down an inclined plane from rest and undergoes slipping. The coordinate system has  $x$  in the direction down the inclined plane and  $y$  upward perpendicular to the plane. The free-body diagram shows the normal force, kinetic friction force, and the components of the weight  $m \vec{g}$ .

The sum of the forces in the  $y$ -direction is zero, so the friction force is now  $f_k = \mu_k N = \mu_k mg \cos \theta$ .

Newton's second law in the  $x$ -direction becomes

$$\begin{aligned} \sum F_x &= ma_x, \\ mg \sin \theta - \mu_k mg \cos \theta &= m(a_{\text{CM}})_x, \end{aligned}$$

or

$$(a_{\text{CM}})_x = g(\sin \theta - \mu_k \cos \theta).$$

The friction force provides the only torque about the axis through the center of mass, so Newton's second law of rotation becomes

$$\begin{aligned} \sum \tau_{\text{CM}} &= I_{\text{CM}} \alpha, \\ f_k r &= I_{\text{CM}} \alpha = \frac{1}{2} mr^2 \alpha. \end{aligned}$$

Solving for  $\alpha$ , we have

$$\alpha = \frac{2f_k}{mr} = \frac{2\mu_k g \cos \theta}{r}.$$

### Significance

We write the linear and angular accelerations in terms of the coefficient of kinetic friction. The linear acceleration is the same as that found for an object sliding down an inclined plane with kinetic friction. The angular acceleration about the axis of rotation is linearly proportional to the normal force, which depends on the cosine of the angle of inclination. As  $\theta \rightarrow 90^\circ$ , this force goes to zero, and, thus, the angular acceleration goes to zero.

## Conservation of Mechanical Energy in Rolling Motion

In the preceding chapter, we introduced rotational kinetic energy. Any rolling object carries rotational kinetic energy, as well as translational kinetic energy and potential energy if the system requires. Including the gravitational potential energy, the total mechanical energy of an object rolling is

$$E_T = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2 + mgh.$$

In the absence of any nonconservative forces that would take energy out of the system in the form of heat, the total energy of a rolling object without slipping is conserved and is constant throughout the motion. Examples where energy is not conserved are a rolling object that is slipping, production of heat as a result of kinetic friction, and a rolling object

encountering air resistance.

You may ask why a rolling object that is not slipping conserves energy, since the static friction force is nonconservative. The answer can be found by referring back to **Figure 11.3**. Point  $P$  in contact with the surface is at rest with respect to the surface. Therefore, its infinitesimal displacement  $d\vec{r}$  with respect to the surface is zero, and the incremental work done by the static friction force is zero. We can apply energy conservation to our study of rolling motion to bring out some interesting results.

### Example 11.3

#### Curiosity Rover

The *Curiosity* rover, shown in **Figure 11.8**, was deployed on Mars on August 6, 2012. The wheels of the rover have a radius of 25 cm. Suppose astronauts arrive on Mars in the year 2050 and find the now-inoperative *Curiosity* on the side of a basin. While they are dismantling the rover, an astronaut accidentally loses a grip on one of the wheels, which rolls without slipping down into the bottom of the basin 25 meters below. If the wheel has a mass of 5 kg, what is its velocity at the bottom of the basin?



**Figure 11.8** The NASA Mars Science Laboratory rover *Curiosity* during testing on June 3, 2011. The location is inside the Spacecraft Assembly Facility at NASA's Jet Propulsion Laboratory in Pasadena, California. (credit: NASA/JPL-Caltech)

#### Strategy

We use mechanical energy conservation to analyze the problem. At the top of the hill, the wheel is at rest and has only potential energy. At the bottom of the basin, the wheel has rotational and translational kinetic energy, which must be equal to the initial potential energy by energy conservation. Since the wheel is rolling without slipping, we use the relation  $v_{\text{CM}} = r\omega$  to relate the translational variables to the rotational variables in the energy conservation equation. We then solve for the velocity. From **Figure 11.8**, we see that a hollow cylinder

is a good approximation for the wheel, so we can use this moment of inertia to simplify the calculation.

### Solution

Energy at the top of the basin equals energy at the bottom:

$$mgh = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2.$$

The known quantities are  $I_{\text{CM}} = mr^2$ ,  $r = 0.25$  m, and  $h = 25.0$  m.

We rewrite the energy conservation equation eliminating  $\omega$  by using  $\omega = \frac{v_{\text{CM}}}{r}$ . We have

$$mgh = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}mr^2 \frac{v_{\text{CM}}^2}{r^2}$$

or

$$gh = \frac{1}{2}v_{\text{CM}}^2 + \frac{1}{2}v_{\text{CM}}^2 \Rightarrow v_{\text{CM}} = \sqrt{gh}.$$

On Mars, the acceleration of gravity is  $3.71$  m/s<sup>2</sup>, which gives the magnitude of the velocity at the bottom of the basin as

$$v_{\text{CM}} = \sqrt{(3.71 \text{ m/s}^2)25.0 \text{ m}} = 9.63 \text{ m/s}.$$

### Significance

This is a fairly accurate result considering that Mars has very little atmosphere, and the loss of energy due to air resistance would be minimal. The result also assumes that the terrain is smooth, such that the wheel wouldn't encounter rocks and bumps along the way.

Also, in this example, the kinetic energy, or energy of motion, is equally shared between linear and rotational motion. If we look at the moments of inertia in **Figure 10.20**, we see that the hollow cylinder has the largest moment of inertia for a given radius and mass. If the wheels of the rover were solid and approximated by solid cylinders, for example, there would be more kinetic energy in linear motion than in rotational motion. This would give the wheel a larger linear velocity than the hollow cylinder approximation. Thus, the solid cylinder would reach the bottom of the basin faster than the hollow cylinder.

## 11.2 | Angular Momentum

### Learning Objectives

By the end of this section, you will be able to:

- Describe the vector nature of angular momentum
- Find the total angular momentum and torque about a designated origin of a system of particles
- Calculate the angular momentum of a rigid body rotating about a fixed axis
- Calculate the torque on a rigid body rotating about a fixed axis
- Use conservation of angular momentum in the analysis of objects that change their rotation rate

Why does Earth keep on spinning? What started it spinning to begin with? Why doesn't Earth's gravitational attraction not bring the Moon crashing in toward Earth? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster?

The answer to these questions is that just as the total linear motion (momentum) in the universe is conserved, so is the total rotational motion conserved. We call the total rotational motion angular momentum, the rotational counterpart to linear momentum. In this chapter, we first define and then explore angular momentum from a variety of viewpoints. First,



however, we investigate the angular momentum of a single particle. This allows us to develop angular momentum for a system of particles and for a rigid body.

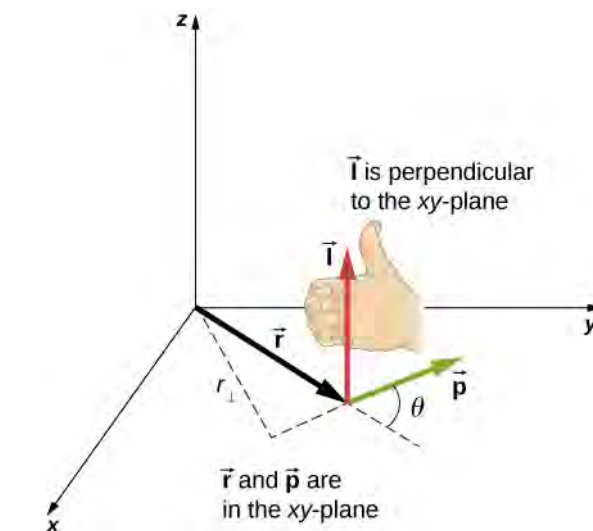
## Angular Momentum of a Single Particle

**Figure 11.9** shows a particle at a position  $\vec{r}$  with linear momentum  $\vec{p} = m\vec{v}$  with respect to the origin. Even if the particle is not rotating about the origin, we can still define an angular momentum in terms of the position vector and the linear momentum.

### Angular Momentum of a Particle

The **angular momentum**  $\vec{l}$  of a particle is defined as the cross-product of  $\vec{r}$  and  $\vec{p}$ , and is perpendicular to the plane containing  $\vec{r}$  and  $\vec{p}$ :

$$\vec{l} = \vec{r} \times \vec{p}. \quad (11.5)$$



**Figure 11.9** In three-dimensional space, the position vector  $\vec{r}$  locates a particle in the  $xy$ -plane with linear momentum  $\vec{p}$ . The angular momentum with respect to the origin is  $\vec{l} = \vec{r} \times \vec{p}$ , which is in the  $z$ -direction. The direction of  $\vec{l}$  is given by the right-hand rule, as shown.

The intent of choosing the direction of the angular momentum to be perpendicular to the plane containing  $\vec{r}$  and  $\vec{p}$  is similar to choosing the direction of torque to be perpendicular to the plane of  $\vec{r}$  and  $\vec{F}$ , as discussed in **Fixed-Axis Rotation**. The magnitude of the angular momentum is found from the definition of the cross-product,

$$l = rp \sin \theta,$$

where  $\theta$  is the angle between  $\vec{r}$  and  $\vec{p}$ . The units of angular momentum are  $\text{kg} \cdot \text{m}^2/\text{s}$ .

As with the definition of torque, we can define a lever arm  $r_{\perp}$  that is the perpendicular distance from the momentum vector  $\vec{p}$  to the origin,  $r_{\perp} = r \sin \theta$ . With this definition, the magnitude of the angular momentum becomes

$$l = r_{\perp} p = r_{\perp} mv.$$

We see that if the direction of  $\vec{\mathbf{p}}$  is such that it passes through the origin, then  $\theta = 0$ , and the angular momentum is zero because the lever arm is zero. In this respect, the magnitude of the angular momentum depends on the choice of origin. If we take the time derivative of the angular momentum, we arrive at an expression for the torque on the particle:

$$\frac{d\vec{\mathbf{L}}}{dt} = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{v}} \times m\vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}.$$

Here we have used the definition of  $\vec{\mathbf{p}}$  and the fact that a vector crossed into itself is zero. From Newton's second law,  $\frac{d\vec{\mathbf{p}}}{dt} = \sum \vec{\mathbf{F}}$ , the net force acting on the particle, and the definition of the net torque, we can write

$$\frac{d\vec{\mathbf{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}. \quad (11.6)$$

Note the similarity with the linear result of Newton's second law,  $\frac{d\vec{\mathbf{p}}}{dt} = \sum \vec{\mathbf{F}}$ . The following problem-solving strategy can serve as a guideline for calculating the angular momentum of a particle.

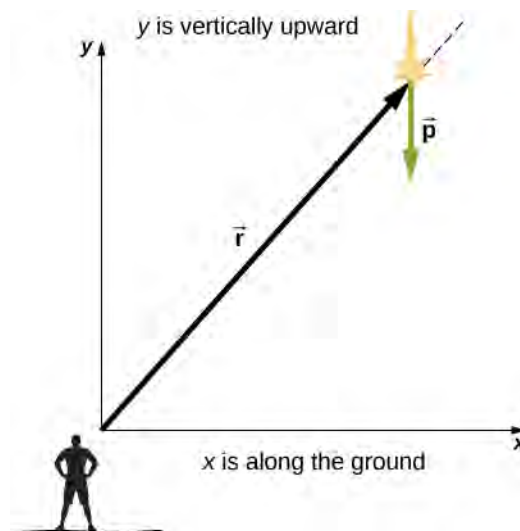
### Problem-Solving Strategy: Angular Momentum of a Particle

1. Choose a coordinate system about which the angular momentum is to be calculated.
2. Write down the radius vector to the point particle in unit vector notation.
3. Write the linear momentum vector of the particle in unit vector notation.
4. Take the cross product  $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$  and use the right-hand rule to establish the direction of the angular momentum vector.
5. See if there is a time dependence in the expression of the angular momentum vector. If there is, then a torque exists about the origin, and use  $\frac{d\vec{\mathbf{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}$  to calculate the torque. If there is no time dependence in the expression for the angular momentum, then the net torque is zero.

## Example 11.4

### Angular Momentum and Torque on a Meteor

A meteor enters Earth's atmosphere (**Figure 11.10**) and is observed by someone on the ground before it burns up in the atmosphere. The vector  $\vec{\mathbf{r}} = 25 \text{ km } \hat{\mathbf{i}} + 25 \text{ km } \hat{\mathbf{j}}$  gives the position of the meteor with respect to the observer. At the instant the observer sees the meteor, it has linear momentum  $\vec{\mathbf{p}} = 15.0 \text{ kg}(-2.0 \text{ km/s } \hat{\mathbf{j}})$ , and it is accelerating at a constant  $2.0 \text{ m/s}^2(-\hat{\mathbf{j}})$  along its path, which for our purposes can be taken as a straight line. (a) What is the angular momentum of the meteor about the origin, which is at the location of the observer? (b) What is the torque on the meteor about the origin?



**Figure 11.10** An observer on the ground sees a meteor at position  $\vec{r}$  with linear momentum  $\vec{p}$ .

### Strategy

We resolve the acceleration into  $x$ - and  $y$ -components and use the kinematic equations to express the velocity as a function of acceleration and time. We insert these expressions into the linear momentum and then calculate the angular momentum using the cross-product. Since the position and momentum vectors are in the  $xy$ -plane, we expect the angular momentum vector to be along the  $z$ -axis. To find the torque, we take the time derivative of the angular momentum.

### Solution

The meteor is entering Earth's atmosphere at an angle of  $90.0^\circ$  below the horizontal, so the components of the acceleration in the  $x$ - and  $y$ -directions are

$$a_x = 0, \quad a_y = -2.0 \text{ m/s}^2.$$

We write the velocities using the kinematic equations.

$$v_x = 0, \quad v_y = -2.0 \times 10^3 \text{ m/s} - (2.0 \text{ m/s}^2)t.$$

- a. The angular momentum is

$$\begin{aligned} \vec{\mathbf{I}} &= \vec{\mathbf{r}} \times \vec{\mathbf{p}} = (25.0 \text{ km} \hat{\mathbf{i}} + 25.0 \text{ km} \hat{\mathbf{j}}) \times 15.0 \text{ kg}(0 \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}) \\ &= 15.0 \text{ kg}[25.0 \text{ km}(v_y) \hat{\mathbf{k}}] \\ &= 15.0 \text{ kg}[2.50 \times 10^4 \text{ m}(-2.0 \times 10^3 \text{ m/s} - (2.0 \text{ m/s}^2)t) \hat{\mathbf{k}}]. \end{aligned}$$

At  $t = 0$ , the angular momentum of the meteor about the origin is

$$\vec{\mathbf{I}}_0 = 15.0 \text{ kg}[2.50 \times 10^4 \text{ m}(-2.0 \times 10^3 \text{ m/s}) \hat{\mathbf{k}}] = 7.50 \times 10^8 \text{ kg} \cdot \text{m}^2/\text{s}(-\hat{\mathbf{k}}).$$

This is the instant that the observer sees the meteor.

- b. To find the torque, we take the time derivative of the angular momentum. Taking the time derivative of  $\vec{\mathbf{I}}$  as a function of time, which is the second equation immediately above, we have

$$\frac{d\vec{\mathbf{I}}}{dt} = -15.0 \text{ kg}(2.50 \times 10^4 \text{ m})(2.0 \text{ m/s}^2) \hat{\mathbf{k}}.$$

Then, since  $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$ , we have

$$\sum \vec{\tau} = -7.5 \times 10^5 \text{ N} \cdot \text{m} \hat{\mathbf{k}}.$$

The units of torque are given as newton-meters, not to be confused with joules. As a check, we note that the lever arm is the x-component of the vector  $\vec{\mathbf{r}}$  in **Figure 11.10** since it is perpendicular to the force acting on the meteor, which is along its path. By Newton's second law, this force is

$$\vec{\mathbf{F}} = ma(-\hat{\mathbf{j}}) = 15.0 \text{ kg}(2.0 \text{ m/s}^2)(-\hat{\mathbf{j}}) = 30.0 \text{ kg} \cdot \text{m/s}^2(-\hat{\mathbf{j}}).$$

The lever arm is

$$\vec{\mathbf{r}}_{\perp} = 2.5 \times 10^4 \text{ m} \hat{\mathbf{i}}.$$


Thus, the torque is

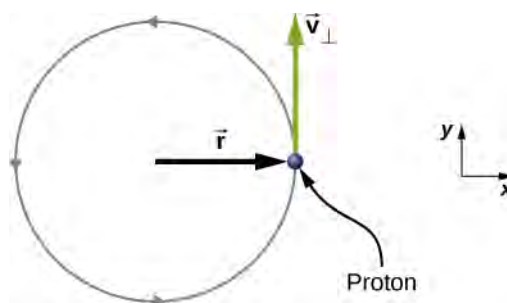
$$\begin{aligned} \sum \vec{\tau} &= \vec{\mathbf{r}}_{\perp} \times \vec{\mathbf{F}} = (2.5 \times 10^4 \text{ m} \hat{\mathbf{i}}) \times (-30.0 \text{ kg} \cdot \text{m/s}^2 \hat{\mathbf{j}}), \\ &= 7.5 \times 10^5 \text{ N} \cdot \text{m}(-\hat{\mathbf{k}}). \end{aligned}$$

### Significance

Since the meteor is accelerating downward toward Earth, its radius and velocity vector are changing. Therefore, since  $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ , the angular momentum is changing as a function of time. The torque on the meteor about the origin, however, is constant, because the lever arm  $\vec{\mathbf{r}}_{\perp}$  and the force on the meteor are constants.

This example is important in that it illustrates that the angular momentum depends on the choice of origin about which it is calculated. The methods used in this example are also important in developing angular momentum for a system of particles and for a rigid body.

-  **11.2 Check Your Understanding** A proton spiraling around a magnetic field executes circular motion in the plane of the paper, as shown below. The circular path has a radius of 0.4 m and the proton has velocity  $4.0 \times 10^6 \text{ m/s}$ . What is the angular momentum of the proton about the origin?



## Angular Momentum of a System of Particles

The angular momentum of a system of particles is important in many scientific disciplines, one being astronomy. Consider a spiral galaxy, a rotating island of stars like our own Milky Way. The individual stars can be treated as point particles, each of which has its own angular momentum. The vector sum of the individual angular momenta give the total angular momentum of the galaxy. In this section, we develop the tools with which we can calculate the total angular momentum of a system of particles.

In the preceding section, we introduced the angular momentum of a single particle about a designated origin. The expression

for this angular momentum is  $\vec{L} = \vec{r} \times \vec{p}$ , where the vector  $\vec{r}$  is from the origin to the particle, and  $\vec{p}$  is the particle's linear momentum. If we have a system of  $N$  particles, each with position vector from the origin given by  $\vec{r}_i$  and each having momentum  $\vec{p}_i$ , then the total angular momentum of the system of particles about the origin is the vector sum of the individual angular momenta about the origin. That is,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N. \quad (11.7)$$

Similarly, if particle  $i$  is subject to a net torque  $\vec{\tau}_i$  about the origin, then we can find the net torque about the origin due to the system of particles by differentiating **Equation 11.7**:

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{L}_i}{dt} = \sum_i \vec{\tau}_i.$$

The sum of the individual torques produces a net external torque on the system, which we designate  $\sum \vec{\tau}$ . Thus,

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}. \quad (11.8)$$

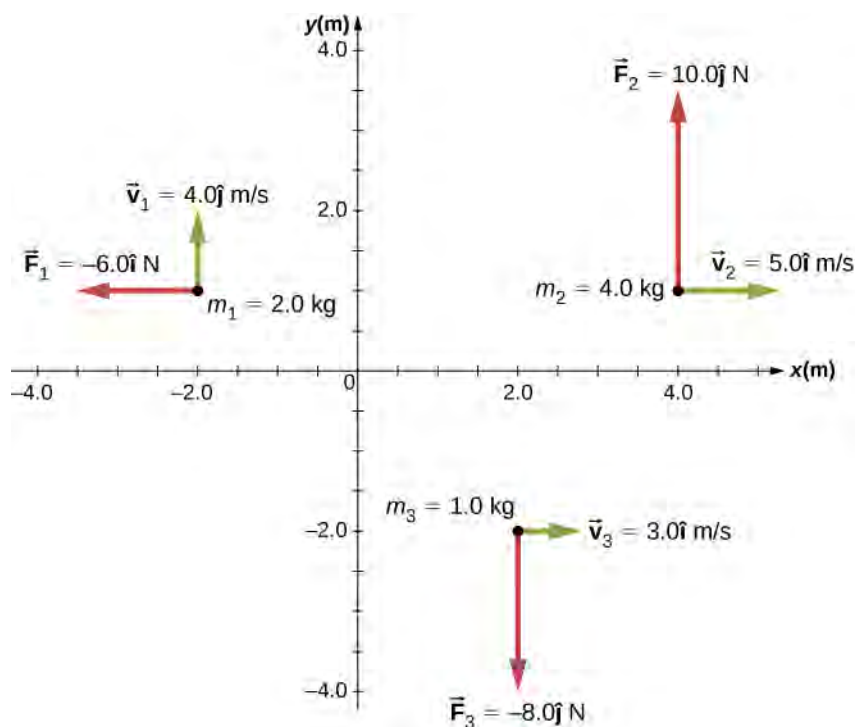
**Equation 11.8** states that *the rate of change of the total angular momentum of a system is equal to the net external torque acting on the system when both quantities are measured with respect to a given origin.* **Equation 11.8** can be applied to any system that has net angular momentum, including rigid bodies, as discussed in the next section.

## Example 11.5

### Angular Momentum of Three Particles

Referring to **Figure 11.11(a)**, determine the total angular momentum due to the three particles about the origin.

(b) What is the rate of change of the angular momentum?



**Figure 11.11** Three particles in the  $xy$ -plane with different position and momentum vectors.

### Strategy

Write down the position and momentum vectors for the three particles. Calculate the individual angular momenta and add them as vectors to find the total angular momentum. Then do the same for the torques.

### Solution

a. Particle 1:  $\vec{r}_1 = -2.0 \text{ m } \hat{i} + 1.0 \text{ m } \hat{j}$ ,  $\vec{p}_1 = 2.0 \text{ kg}(4.0 \text{ m/s } \hat{j}) = 8.0 \text{ kg} \cdot \text{m/s } \hat{j}$ ,

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{p}_1 = -16.0 \text{ kg} \cdot \text{m}^2/\text{s} \hat{k}.$$

Particle 2:  $\vec{r}_2 = 4.0 \text{ m } \hat{i} + 1.0 \text{ m } \hat{j}$ ,  $\vec{p}_2 = 4.0 \text{ kg}(5.0 \text{ m/s } \hat{i}) = 20.0 \text{ kg} \cdot \text{m/s } \hat{i}$ ,

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{p}_2 = -20.0 \text{ kg} \cdot \text{m}^2/\text{s} \hat{k}.$$

Particle 3:  $\vec{r}_3 = 2.0 \text{ m } \hat{i} - 2.0 \text{ m } \hat{j}$ ,  $\vec{p}_3 = 1.0 \text{ kg}(3.0 \text{ m/s } \hat{i}) = 3.0 \text{ kg} \cdot \text{m/s } \hat{i}$ ,

$$\vec{\tau}_3 = \vec{r}_3 \times \vec{p}_3 = 6.0 \text{ kg} \cdot \text{m}^2/\text{s} \hat{k}.$$

We add the individual angular momenta to find the total about the origin:

$$\vec{\tau}_T = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = -30 \text{ kg} \cdot \text{m}^2/\text{s} \hat{k}.$$

b. The individual forces and lever arms are

$$\begin{aligned}\vec{\mathbf{r}}_{1\perp} &= 1.0\text{ m}\hat{\mathbf{j}}, & \vec{\mathbf{F}}_1 &= -6.0\text{ N}\hat{\mathbf{i}}, & \vec{\boldsymbol{\tau}}_1 &= 6.0\text{ N}\cdot\text{m}\hat{\mathbf{k}} \\ \vec{\mathbf{r}}_{2\perp} &= 4.0\text{ m}\hat{\mathbf{i}}, & \vec{\mathbf{F}}_2 &= 10.0\text{ N}\hat{\mathbf{j}}, & \vec{\boldsymbol{\tau}}_2 &= 40.0\text{ N}\cdot\text{m}\hat{\mathbf{k}} \\ \vec{\mathbf{r}}_{3\perp} &= 2.0\text{ m}\hat{\mathbf{i}}, & \vec{\mathbf{F}}_3 &= -8.0\text{ N}\hat{\mathbf{j}}, & \vec{\boldsymbol{\tau}}_3 &= -16.0\text{ N}\cdot\text{m}\hat{\mathbf{k}}.\end{aligned}$$

Therefore:

$$\sum_i \vec{\boldsymbol{\tau}}_i = \vec{\boldsymbol{\tau}}_1 + \vec{\boldsymbol{\tau}}_2 + \vec{\boldsymbol{\tau}}_3 = 30\text{ N}\cdot\text{m}\hat{\mathbf{k}}.$$

### Significance

This example illustrates the superposition principle for angular momentum and torque of a system of particles. Care must be taken when evaluating the radius vectors  $\vec{\mathbf{r}}_i$  of the particles to calculate the angular momenta, and the lever arms,  $\vec{\mathbf{r}}_{i\perp}$  to calculate the torques, as they are completely different quantities.

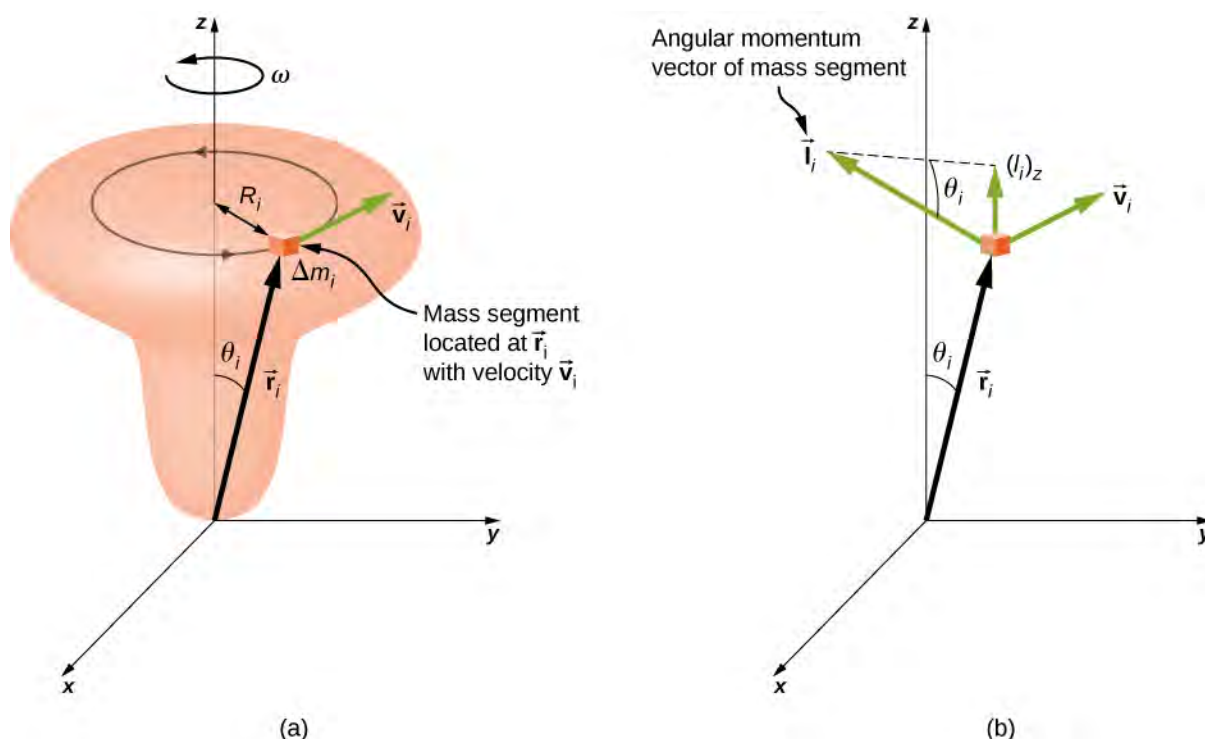
## Angular Momentum of a Rigid Body

We have investigated the angular momentum of a single particle, which we generalized to a system of particles. Now we can use the principles discussed in the previous section to develop the concept of the angular momentum of a rigid body. Celestial objects such as planets have angular momentum due to their spin and orbits around stars. In engineering, anything that rotates about an axis carries angular momentum, such as flywheels, propellers, and rotating parts in engines. Knowledge of the angular momenta of these objects is crucial to the design of the system in which they are a part.

To develop the angular momentum of a rigid body, we model a rigid body as being made up of small mass segments,  $\Delta m_i$ .

In **Figure 11.12**, a rigid body is constrained to rotate about the  $z$ -axis with angular velocity  $\omega$ . All mass segments that make up the rigid body undergo circular motion about the  $z$ -axis with the same angular velocity. Part (a) of the figure shows mass segment  $\Delta m_i$  with position vector  $\vec{\mathbf{r}}_i$  from the origin and radius  $R_i$  to the  $z$ -axis. The magnitude of its tangential velocity is  $v_i = R_i\omega$ . Because the vectors  $\vec{\mathbf{v}}_i$  and  $\vec{\mathbf{r}}_i$  are perpendicular to each other, the magnitude of the angular momentum of this mass segment is

$$l_i = r_i(\Delta m v_i)\sin 90^\circ.$$



**Figure 11.12** (a) A rigid body is constrained to rotate around the z-axis. The rigid body is symmetrical about the z-axis. A mass segment  $\Delta m_i$  is located at position  $\vec{r}_i$ , which makes angle  $\theta_i$  with respect to the z-axis. The circular motion of an infinitesimal mass segment is shown. (b)  $\vec{l}_i$  is the angular momentum of the mass segment and has a component along the z-axis ( $\vec{l}_i$ )<sub>z</sub>.

Using the right-hand rule, the angular momentum vector points in the direction shown in part (b). The sum of the angular momenta of all the mass segments contains components both along and perpendicular to the axis of rotation. Every mass segment has a perpendicular component of the angular momentum that will be cancelled by the perpendicular component of an identical mass segment on the opposite side of the rigid body. Thus, the component along the axis of rotation is the only component that gives a nonzero value when summed over all the mass segments. From part (b), the component of  $\vec{l}_i$  along the axis of rotation is

$$\begin{aligned}(l_i)_z &= l_i \sin \theta_i = (r_i \Delta m_i v_i) \sin \theta_i, \\ &= (r_i \sin \theta_i)(\Delta m_i v_i) = R_i \Delta m_i v_i.\end{aligned}$$

The net angular momentum of the rigid body along the axis of rotation is

$$L = \sum_i (\vec{l}_i)_z = \sum_i R_i \Delta m_i v_i = \sum_i R_i \Delta m_i (R_i \omega) = \omega \sum_i \Delta m_i (R_i)^2.$$

The summation  $\sum_i \Delta m_i (R_i)^2$  is simply the moment of inertia  $I$  of the rigid body about the axis of rotation. For a thin hoop rotating about an axis perpendicular to the plane of the hoop, all of the  $R_i$ 's are equal to  $R$  so the summation reduces to  $R^2 \sum_i \Delta m_i = mR^2$ , which is the moment of inertia for a thin hoop found in **Figure 10.20**. Thus, the magnitude of the angular momentum along the axis of rotation of a rigid body rotating with angular velocity  $\omega$  about the axis is

$$L = I\omega. \quad (11.9)$$

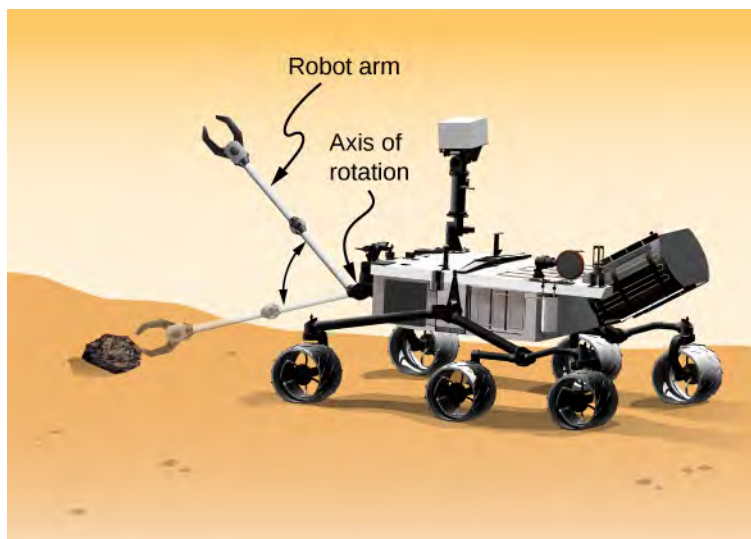


This equation is analogous to the magnitude of the linear momentum  $p = mv$ . The direction of the angular momentum vector is directed along the axis of rotation given by the right-hand rule.

## Example 11.6

### Angular Momentum of a Robot Arm

A robot arm on a Mars rover like *Curiosity* shown in **Figure 11.8** is 1.0 m long and has forceps at the free end to pick up rocks. The mass of the arm is 2.0 kg and the mass of the forceps is 1.0 kg. See **Figure 11.13**. The robot arm and forceps move from rest to  $\omega = 0.1\pi$  rad/s in 0.1 s. It rotates down and picks up a Mars rock that has mass 1.5 kg. The axis of rotation is the point where the robot arm connects to the rover. (a) What is the angular momentum of the robot arm by itself about the axis of rotation after 0.1 s when the arm has stopped accelerating? (b) What is the angular momentum of the robot arm when it has the Mars rock in its forceps and is rotating upwards? (c) When the arm does not have a rock in the forceps, what is the torque about the point where the arm connects to the rover when it is accelerating from rest to its final angular velocity?



**Figure 11.13** A robot arm on a Mars rover swings down and picks up a Mars rock. (credit: modification of work by NASA/JPL-Caltech)

### Strategy

We use **Equation 11.9** to find angular momentum in the various configurations. When the arm is rotating downward, the right-hand rule gives the angular momentum vector directed out of the page, which we will call the positive  $z$ -direction. When the arm is rotating upward, the right-hand rule gives the direction of the angular momentum vector into the page or in the negative  $z$ -direction. The moment of inertia is the sum of the individual moments of inertia. The arm can be approximated with a solid rod, and the forceps and Mars rock can be approximated as point masses located at a distance of 1 m from the origin. For part (c), we use Newton's second law of motion for rotation to find the torque on the robot arm.

### Solution

- a. Writing down the individual moments of inertia, we have

$$\text{Robot arm: } I_R = \frac{1}{3}m_R r^2 = \frac{1}{3}(2.00 \text{ kg})(1.00 \text{ m})^2 = \frac{2}{3} \text{ kg} \cdot \text{m}^2.$$

$$\text{Forceps: } I_F = m_F r^2 = (1.0 \text{ kg})(1.0 \text{ m})^2 = 1.0 \text{ kg} \cdot \text{m}^2.$$

$$\text{Mars rock: } I_{MR} = m_{MR} r^2 = (1.5 \text{ kg})(1.0 \text{ m})^2 = 1.5 \text{ kg} \cdot \text{m}^2.$$

Therefore, without the Mars rock, the total moment of inertia is

$$I_{\text{Total}} = I_R + I_F = 1.67 \text{ kg} \cdot \text{m}^2$$

and the magnitude of the angular momentum is

$$L = I\omega = 1.67 \text{ kg} \cdot \text{m}^2(0.1\pi \text{ rad/s}) = 0.17\pi \text{ kg} \cdot \text{m}^2/\text{s}.$$

The angular momentum vector is directed out of the page in the  $\hat{\mathbf{k}}$  direction since the robot arm is rotating counterclockwise.

- b. We must include the Mars rock in the calculation of the moment of inertia, so we have

$$I_{\text{Total}} = I_{\text{R}} + I_{\text{F}} + I_{\text{MR}} = 3.17 \text{ kg} \cdot \text{m}^2$$

and

$$L = I\omega = 3.17 \text{ kg} \cdot \text{m}^2(0.1\pi \text{ rad/s}) = 0.32\pi \text{ kg} \cdot \text{m}^2/\text{s}.$$

Now the angular momentum vector is directed into the page in the  $-\hat{\mathbf{k}}$  direction, by the right-hand rule, since the robot arm is now rotating clockwise.

- c. We find the torque when the arm does not have the rock by taking the derivative of the angular momentum using **Equation 11.8**  $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$ . But since  $L = I\omega$ , and understanding that the direction of the angular momentum and torque vectors are along the axis of rotation, we can suppress the vector notation and find

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha = \sum \tau,$$

which is Newton's second law for rotation. Since  $\alpha = \frac{0.1\pi \text{ rad/s}}{0.1 \text{ s}} = \pi \text{ rad/s}^2$ , we can calculate the net torque:

$$\sum \tau = I\alpha = 1.67 \text{ kg} \cdot \text{m}^2(\pi \text{ rad/s}^2) = 1.67\pi \text{ N} \cdot \text{m}.$$

### Significance

The angular momentum in (a) is less than that of (b) due to the fact that the moment of inertia in (b) is greater than (a), while the angular velocity is the same.



**11.3 Check Your Understanding** Which has greater angular momentum: a solid sphere of mass  $m$  rotating at a constant angular frequency  $\omega_0$  about the  $z$ -axis, or a solid cylinder of same mass and rotation rate about the  $z$ -axis?



Visit the **University of Colorado's Interactive Simulation of Angular Momentum** (<https://openstaxcollege.org//21angmomintsim>) to learn more about angular momentum.

## 11.3 | Conservation of Angular Momentum

### Learning Objectives

By the end of this section, you will be able to:

- Apply conservation of angular momentum to determine the angular velocity of a rotating system in which the moment of inertia is changing
- Explain how the rotational kinetic energy changes when a system undergoes changes in both moment of inertia and angular velocity

So far, we have looked at the angular momentum of systems consisting of point particles and rigid bodies. We have also analyzed the torques involved, using the expression that relates the external net torque to the change in angular momentum, **Equation 11.8**. Examples of systems that obey this equation include a freely spinning bicycle tire that slows over time due to torque arising from friction, or the slowing of Earth's rotation over millions of years due to frictional forces exerted on tidal deformations.

However, suppose there is no net external torque on the system,  $\sum \vec{\tau} = 0$ . In this case, **Equation 11.8** becomes the **law of conservation of angular momentum**.

### Law of Conservation of Angular Momentum

The angular momentum of a system of particles around a point in a fixed inertial reference frame is conserved if there is no net external torque around that point:

$$\frac{d\vec{L}}{dt} = 0 \quad (11.10)$$

or

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \text{constant}. \quad (11.11)$$

Note that the *total* angular momentum  $\vec{L}$  is conserved. Any of the individual angular momenta can change as long as their sum remains constant. This law is analogous to linear momentum being conserved when the external force on a system is zero.

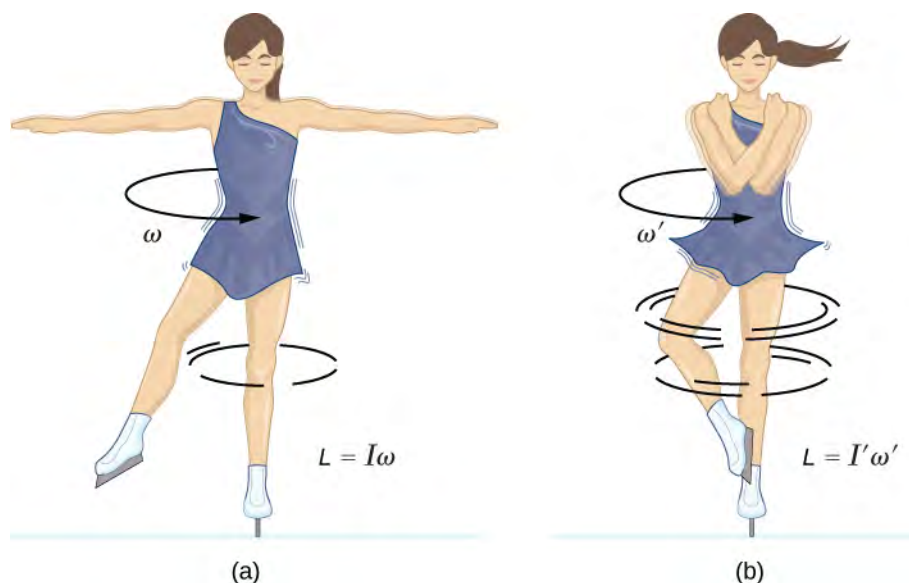
As an example of conservation of angular momentum, **Figure 11.14** shows an ice skater executing a spin. The net torque on her is very close to zero because there is relatively little friction between her skates and the ice. Also, the friction is exerted very close to the pivot point. Both  $|\vec{r}|$  and  $|\vec{F}|$  are small, so  $|\vec{\tau}|$  is negligible. Consequently, she can spin for quite some time. She can also increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

$$L' = L$$

or

$$I' \omega' = I \omega,$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because  $I'$  is smaller, the angular velocity  $\omega'$  must increase to keep the angular momentum constant.



**Figure 11.14** (a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. (b) Her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

It is interesting to see how the rotational kinetic energy of the skater changes when she pulls her arms in. Her initial rotational energy is

$$K_{\text{Rot}} = \frac{1}{2}I\omega^2,$$

whereas her final rotational energy is

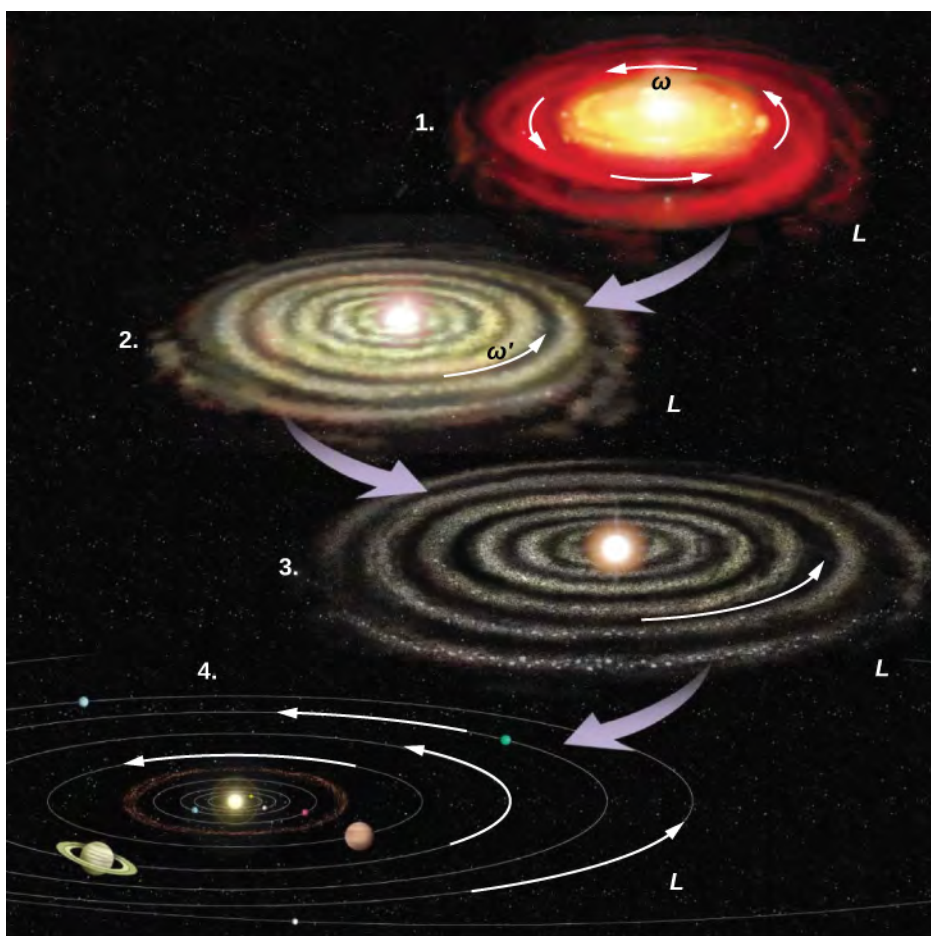
$$K'_{\text{Rot}} = \frac{1}{2}I'(\omega')^2.$$

Since  $I'\omega' = I\omega$ , we can substitute for  $\omega'$  and find

$$K'_{\text{Rot}} = \frac{1}{2}I'(\omega')^2 = \frac{1}{2}I'\left(\frac{I\omega}{I'}\right)^2 = \frac{1}{2}I\omega^2\left(\frac{I}{I'}\right) = K_{\text{Rot}}\left(\frac{I}{I'}\right).$$

Because her moment of inertia has decreased,  $I' < I$ , her final rotational kinetic energy has increased. The source of this additional rotational kinetic energy is the work required to pull her arms inward. Note that the skater's arms do not move in a perfect circle—they spiral inward. This work causes an increase in the rotational kinetic energy, while her angular momentum remains constant. Since she is in a frictionless environment, no energy escapes the system. Thus, if she were to extend her arms to their original positions, she would rotate at her original angular velocity and her kinetic energy would return to its original value.

The solar system is another example of how conservation of angular momentum works in our universe. Our solar system was born from a huge cloud of gas and dust that initially had rotational energy. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result of conservation of angular momentum (**Figure 11.15**).



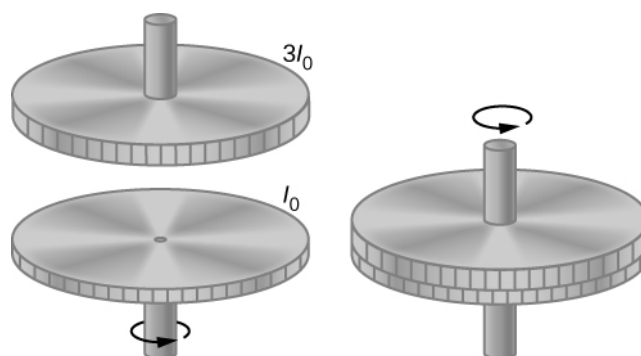
**Figure 11.15** The solar system coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud. (credit: modification of work by NASA)

We continue our discussion with an example that has applications to engineering.

### Example 11.7

#### Coupled Flywheels

A flywheel rotates without friction at an angular velocity  $\omega_0 = 600 \text{ rev/min}$  on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it (**Figure 11.16**). Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together. (a) Use the law of conservation of angular momentum to determine the angular velocity  $\omega$  of the combination. (b) What fraction of the initial kinetic energy is lost in the coupling of the flywheels?



**Figure 11.16** Two flywheels are coupled and rotate together.

### Strategy

Part (a) is straightforward to solve for the angular velocity of the coupled system. We use the result of (a) to compare the initial and final kinetic energies of the system in part (b).

### Solution

a. No external torques act on the system. The force due to friction produces an internal torque, which does not affect the angular momentum of the system. Therefore conservation of angular momentum gives

$$I_0 \omega_0 = (I_0 + 3I_0) \omega,$$

$$\omega = \frac{1}{4} \omega_0 = 150 \text{ rev/min} = 15.7 \text{ rad/s}.$$

b. Before contact, only one flywheel is rotating. The rotational kinetic energy of this flywheel is the initial rotational kinetic energy of the system,  $\frac{1}{2} I_0 \omega_0^2$ . The final kinetic energy is  $\frac{1}{2} (4I_0) \omega^2 = \frac{1}{2} (4I_0) \left(\frac{\omega_0}{4}\right)^2 = \frac{1}{8} I_0 \omega_0^2$ .

Therefore, the ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{\frac{1}{8} I_0 \omega_0^2}{\frac{1}{2} I_0 \omega_0^2} = \frac{1}{4}.$$

Thus, 3/4 of the initial kinetic energy is lost to the coupling of the two flywheels.

### Significance

Since the rotational inertia of the system increased, the angular velocity decreased, as expected from the law of conservation of angular momentum. In this example, we see that the final kinetic energy of the system has decreased, as energy is lost to the coupling of the flywheels. Compare this to the example of the skater in **Figure 11.14** doing work to bring her arms inward and adding rotational kinetic energy.



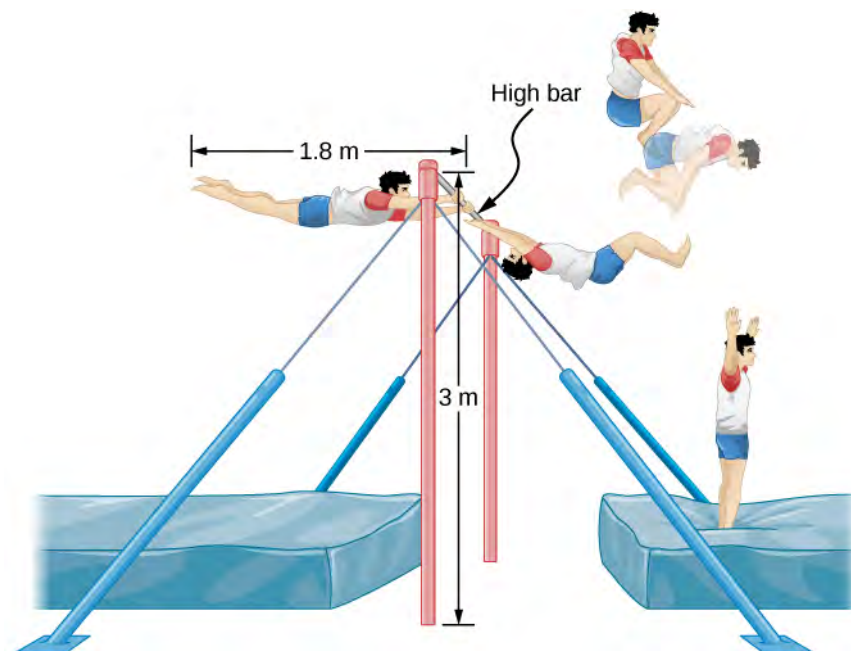
**11.4 Check Your Understanding** A merry-go-round at a playground is rotating at 4.0 rev/min. Three children jump on and increase the moment of inertia of the merry-go-round/children rotating system by 25%. What is the new rotation rate?

## Example 11.8

### Dismount from a High Bar

An 80.0-kg gymnast dismounts from a high bar. He starts the dismount at full extension, then tucks to complete a number of revolutions before landing. His moment of inertia when fully extended can be approximated as a rod of length 1.8 m and when in the tuck a rod of half that length. If his rotation rate at full extension is 1.0 rev/s and he enters the tuck when his center of mass is at 3.0 m height moving horizontally to the floor, how many revolutions

can he execute if he comes out of the tuck at 1.8 m height? See **Figure 11.17**.



**Figure 11.17** A gymnast dismounts from a high bar and executes a number of revolutions in the tucked position before landing upright.

### Strategy

Using conservation of angular momentum, we can find his rotation rate when in the tuck. Using the equations of kinematics, we can find the time interval from a height of 3.0 m to 1.8 m. Since he is moving horizontally with respect to the ground, the equations of free fall simplify. This will allow the number of revolutions that can be executed to be calculated. Since we are using a ratio, we can keep the units as rev/s and don't need to convert to radians/s.

### Solution

The moment of inertia at full extension is  $I_0 = \frac{1}{12}mL^2 = \frac{1}{12}80.0 \text{ kg}(1.8 \text{ m})^2 = 21.6 \text{ kg} \cdot \text{m}^2$ .

The moment of inertia in the tuck is  $I_f = \frac{1}{12}mL_f^2 = \frac{1}{12}80.0 \text{ kg}(0.9 \text{ m})^2 = 5.4 \text{ kg} \cdot \text{m}^2$ .

Conservation of angular momentum:  $I_f \omega_f = I_0 \omega_0 \Rightarrow \omega_f = \frac{I_0 \omega_0}{I_f} = \frac{21.6 \text{ kg} \cdot \text{m}^2(1.0 \text{ rev/s})}{5.4 \text{ kg} \cdot \text{m}^2} = 4.0 \text{ rev/s}$ .

Time interval in the tuck:  $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(3.0 - 1.8)\text{m}}{9.8 \text{ m/s}^2}} = 0.5 \text{ s}$ .

In 0.5 s, he will be able to execute two revolutions at 4.0 rev/s.

### Significance

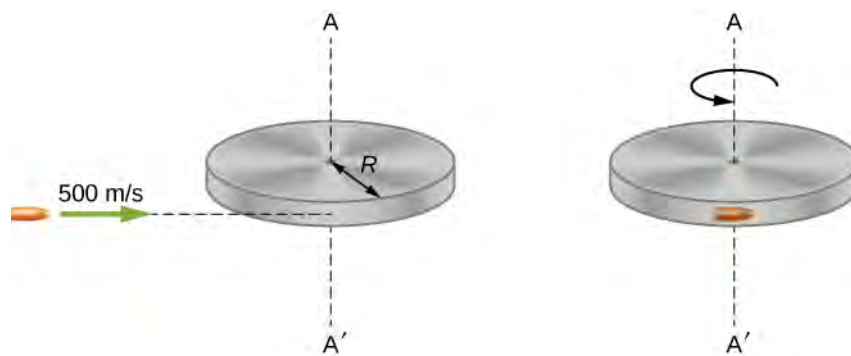
Note that the number of revolutions he can complete will depend on how long he is in the air. In the problem, he is exiting the high bar horizontally to the ground. He could also exit at an angle with respect to the ground, giving him more or less time in the air depending on the angle, positive or negative, with respect to the ground. Gymnasts must take this into account when they are executing their dismounts.



## Example 11.9

### Conservation of Angular Momentum of a Collision

A bullet of mass  $m = 2.0 \text{ g}$  is moving horizontally with a speed of  $500.0 \text{ m/s}$ . The bullet strikes and becomes embedded in the edge of a solid disk of mass  $M = 3.2 \text{ kg}$  and radius  $R = 0.5 \text{ m}$ . The cylinder is free to rotate around its axis and is initially at rest (Figure 11.18). What is the angular velocity of the disk immediately after the bullet is embedded?



**Figure 11.18** A bullet is fired horizontally and becomes embedded in the edge of a disk that is free to rotate about its vertical axis.

### Strategy

For the system of the bullet and the cylinder, no external torque acts along the vertical axis through the center of the disk. Thus, the angular momentum along this axis is conserved. The initial angular momentum of the bullet is  $mvR$ , which is taken about the rotational axis of the disk the moment before the collision. The initial angular momentum of the cylinder is zero. Thus, the net angular momentum of the system is  $mvR$ . Since angular momentum is conserved, the initial angular momentum of the system is equal to the angular momentum of the bullet embedded in the disk immediately after impact.

### Solution

The initial angular momentum of the system is

$$L_i = mvR.$$

The moment of inertia of the system with the bullet embedded in the disk is

$$I = mR^2 + \frac{1}{2}MR^2 = \left(m + \frac{M}{2}\right)R^2.$$

The final angular momentum of the system is

$$L_f = I\omega_f.$$

Thus, by conservation of angular momentum,  $L_i = L_f$  and

$$mvR = \left(m + \frac{M}{2}\right)R^2\omega_f.$$

Solving for  $\omega_f$ ,

$$\omega_f = \frac{mvR}{\left(m + \frac{M}{2}\right)R^2} = \frac{(2.0 \times 10^{-3} \text{ kg})(500.0 \text{ m/s})}{(2.0 \times 10^{-3} \text{ kg} + 1.6 \text{ kg})(0.50 \text{ m})} = 1.2 \text{ rad/s}.$$

### Significance

The system is composed of both a point particle and a rigid body. Care must be taken when formulating the



angular momentum before and after the collision. Just before impact the angular momentum of the bullet is taken about the rotational axis of the disk.

## 11.4 | Precession of a Gyroscope

### Learning Objectives

By the end of this section, you will be able to:

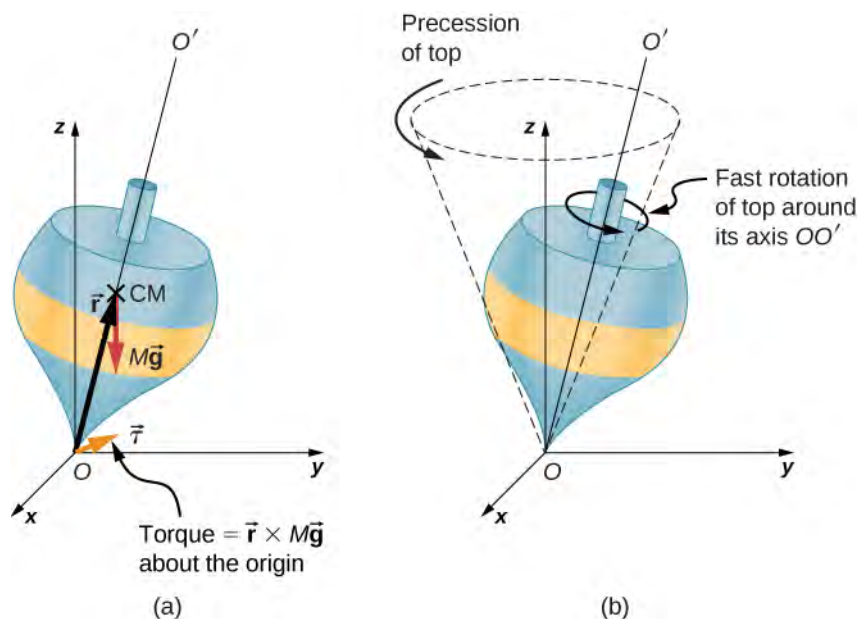
- Describe the physical processes underlying the phenomenon of precession
- Calculate the precessional angular velocity of a gyroscope

**Figure 11.19** shows a gyroscope, defined as a spinning disk in which the axis of rotation is free to assume any orientation. When spinning, the orientation of the spin axis is unaffected by the orientation of the body that encloses it. The body or vehicle enclosing the gyroscope can be moved from place to place and the orientation of the spin axis will remain the same. This makes gyroscopes very useful in navigation, especially where magnetic compasses can't be used, such as in manned and unmanned spacecraft, intercontinental ballistic missiles, unmanned aerial vehicles, and satellites like the Hubble Space Telescope.



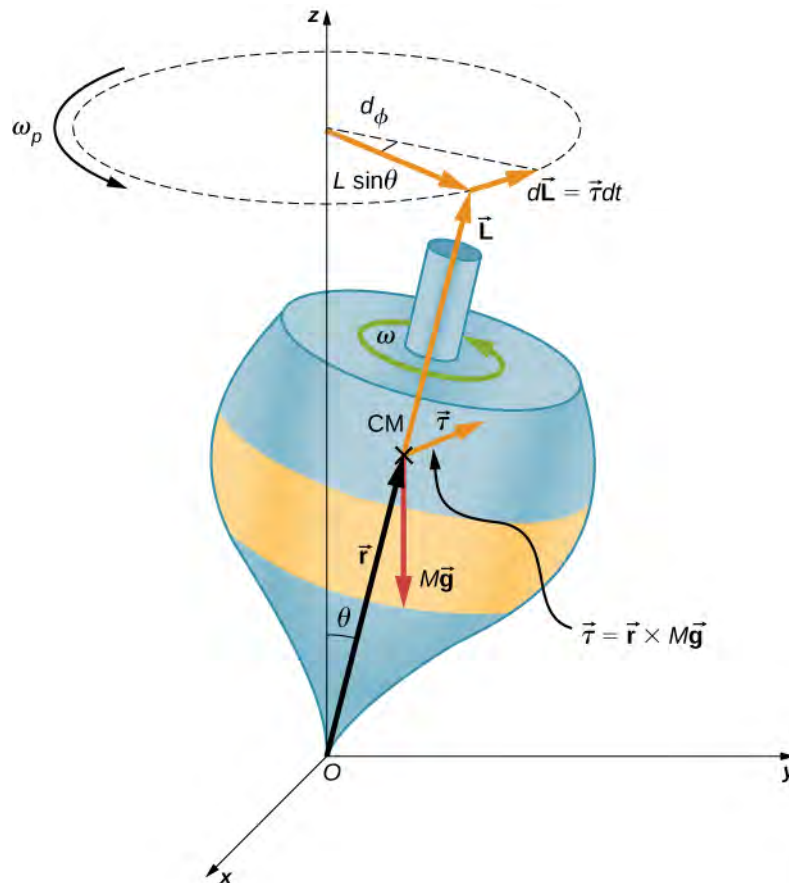
**Figure 11.19** A gyroscope consists of a spinning disk about an axis that is free to assume any orientation.

We illustrate the **precession** of a gyroscope with an example of a top in the next two figures. If the top is placed on a flat surface near the surface of Earth at an angle to the vertical and is not spinning, it will fall over, due to the force of gravity producing a torque acting on its center of mass. This is shown in **Figure 11.20(a)**. However, if the top is spinning on its axis, rather than topple over due to this torque, it precesses about the vertical, shown in part (b) of the figure. This is due to the torque on the center of mass, which provides the change in angular momentum.



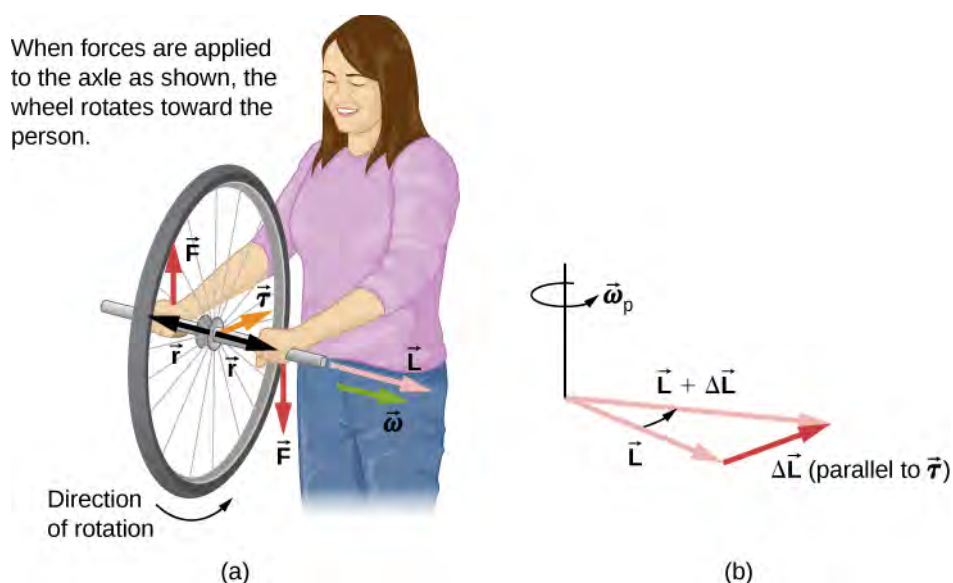
**Figure 11.20** (a) If the top is not spinning, there is a torque  $\vec{\tau} = M \vec{g} \times \vec{r}$  about the origin, and the top falls over. (b) If the top is spinning about its axis  $OO'$ , it doesn't fall over but precesses about the  $z$ -axis.

**Figure 11.21** shows the forces acting on a spinning top. The torque produced is perpendicular to the angular momentum vector. This changes the direction of the angular momentum vector  $\vec{L}$  according to  $d\vec{L} = \vec{\tau} dt$ , but not its magnitude. The top *precesses* around a vertical axis, since the torque is always horizontal and perpendicular to  $\vec{L}$ . If the top is *not* spinning, it acquires angular momentum in the direction of the torque, and it rotates around a horizontal axis, falling over just as we would expect.



**Figure 11.21** The force of gravity acting on the center of mass produces a torque  $\vec{\tau}$  in the direction perpendicular to  $\vec{L}$ . The magnitude of  $\vec{L}$  doesn't change but its direction does, and the top precesses about the z-axis.

We can experience this phenomenon first hand by holding a spinning bicycle wheel and trying to rotate it about an axis perpendicular to the spin axis. As shown in **Figure 11.22**, the person applies forces perpendicular to the spin axis in an attempt to rotate the wheel, but instead, the wheel axis starts to change direction to her left due to the applied torque.



**Figure 11.22** (a) A person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum  $\Delta \vec{L}$  in exactly the same direction. (b) A vector diagram depicting how  $\Delta \vec{L}$  and  $\vec{L}$  add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

We all know how easy it is for a bicycle to tip over when sitting on it at rest. But when riding the bicycle at a good pace, it is harder to tip it over because we must change the angular momentum vector of the spinning wheels.

 View the video on **gyroscope precession** (<https://openstaxcollege.org//21gyrovideo>) for a complete demonstration of precession of the bicycle wheel.

Also, when a spinning disk is put in a box such as a Blu-Ray player, try to move it. It is easy to translate the box in a given direction but difficult to rotate it about an axis perpendicular to the axis of the spinning disk, since we are putting a torque on the box that will cause the angular momentum vector of the spinning disk to precess.

We can calculate the precession rate of the top in **Figure 11.21**. From **Figure 11.21**, we see that the magnitude of the torque is

$$\tau = rMg \sin \theta.$$

Thus,

$$dL = rMg \sin \theta dt.$$

The angle the top precesses through in time  $dt$  is

$$d\phi = \frac{dL}{L \sin \theta} = \frac{rMg \sin \theta}{L \sin \theta} dt = \frac{rMg}{L} dt.$$

The precession angular velocity is  $\omega_P = \frac{d\phi}{dt}$  and from this equation we see that

$$\omega_P = \frac{rMg}{L}. \text{ or, since } L = I\omega,$$

$$\omega_P = \frac{rMg}{I\omega}. \quad (11.12)$$

In this derivation, we assumed that  $\omega_P \ll \omega$ , that is, that the precession angular velocity is much less than the angular velocity of the gyroscope disk. The precession angular velocity adds a small component to the angular momentum along the z-axis. This is seen in a slight bob up and down as the gyroscope precesses, referred to as nutation.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and currently points at Polaris, the North Star. But Earth is slowly precessing (once in about 26,000 years) due to the torque of the Sun and the Moon on its nonspherical shape.

## Example 11.10

### Period of Precession

A gyroscope spins with its tip on the ground and is spinning with negligible frictional resistance. The disk of the gyroscope has mass 0.3 kg and is spinning at 20 rev/s. Its center of mass is 5.0 cm from the pivot and the radius of the disk is 5.0 cm. What is the precessional period of the gyroscope?

### Strategy

We use **Equation 11.12** to find the precessional angular velocity of the gyroscope. This allows us to find the period of precession.

### Solution

The moment of inertia of the disk is

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(0.30 \text{ kg})(0.05 \text{ m})^2 = 3.75 \times 10^{-4} \text{ kg} \cdot \text{m}^2.$$

The angular velocity of the disk is

$$20.0 \text{ rev/s} = 20.0(2\pi) \text{ rad/s} = 125.66 \text{ rad/s}.$$

We can now substitute in **Equation 11.12**. The precessional angular velocity is

$$\omega_P = \frac{rMg}{I\omega} = \frac{(0.05 \text{ m})(0.3 \text{ kg})(9.8 \text{ m/s}^2)}{(3.75 \times 10^{-4} \text{ kg} \cdot \text{m}^2)(125.66 \text{ rad/s})} = 3.12 \text{ rad/s}.$$

The precessional period of the gyroscope is

$$T_P = \frac{2\pi}{3.12 \text{ rad/s}} = 2.0 \text{ s}.$$

### Significance

The precessional angular frequency of the gyroscope, 3.12 rad/s, or about 0.5 rev/s, is much less than the angular velocity 20 rev/s of the gyroscope disk. Therefore, we don't expect a large component of the angular momentum to arise due to precession, and **Equation 11.12** is a good approximation of the precessional angular velocity.



**11.5 Check Your Understanding** A top has a precession frequency of 5.0 rad/s on Earth. What is its precession frequency on the Moon?

## CHAPTER 11 REVIEW

### KEY TERMS

**angular momentum** rotational analog of linear momentum, found by taking the product of moment of inertia and angular velocity

**law of conservation of angular momentum** angular momentum is conserved, that is, the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

**precession** circular motion of the pole of the axis of a spinning object around another axis due to a torque

**rolling motion** combination of rotational and translational motion with or without slipping

### KEY EQUATIONS

Velocity of center of mass of rolling object	$v_{\text{CM}} = R\omega$
Acceleration of center of mass of rolling object	$a_{\text{CM}} = R\alpha$
Displacement of center of mass of rolling object	$d_{\text{CM}} = R\theta$
Acceleration of an object rolling without slipping	$a_{\text{CM}} = \frac{mg \sin \theta}{m + (I_{\text{CM}}/r^2)}$
Angular momentum	$\vec{L} = \vec{r} \times \vec{p}$
Derivative of angular momentum equals torque	$\frac{d\vec{L}}{dt} = \sum \vec{\tau}$
Angular momentum of a system of particles	$\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_N$
For a system of particles, derivative of angular momentum equals torque	$\frac{d\vec{L}}{dt} = \sum \vec{\tau}$
Angular momentum of a rotating rigid body	$L = I\omega$
Conservation of angular momentum	$\frac{d\vec{L}}{dt} = 0$
Conservation of angular momentum	$\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_N = \text{constant}$
Precessional angular velocity	$\omega_P = \frac{rMg}{I\omega}$

## SUMMARY

### 11.1 Rolling Motion

- In rolling motion without slipping, a static friction force is present between the rolling object and the surface. The relations  $v_{\text{CM}} = R\omega$ ,  $a_{\text{CM}} = R\alpha$ , and  $d_{\text{CM}} = R\theta$  all apply, such that the linear velocity, acceleration, and distance of the center of mass are the angular variables multiplied by the radius of the object.
- In rolling motion with slipping, a kinetic friction force arises between the rolling object and the surface. In this case,  $v_{\text{CM}} \neq R\omega$ ,  $a_{\text{CM}} \neq R\alpha$ , and  $d_{\text{CM}} \neq R\theta$ .
- Energy conservation can be used to analyze rolling motion. Energy is conserved in rolling motion without slipping. Energy is not conserved in rolling motion with slipping due to the heat generated by kinetic friction.

## 11.2 Angular Momentum

- The angular momentum  $\vec{L} = \vec{r} \times \vec{p}$  of a single particle about a designated origin is the vector product of the position vector in the given coordinate system and the particle's linear momentum.
- The angular momentum  $\vec{L} = \sum_i \vec{L}_i$  of a system of particles about a designated origin is the vector sum of the individual momenta of the particles that make up the system.
- The net torque on a system about a given origin is the time derivative of the angular momentum about that origin:  $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$ .
- A rigid rotating body has angular momentum  $L = I\omega$  directed along the axis of rotation. The time derivative of the angular momentum  $\frac{dL}{dt} = \sum \tau$  gives the net torque on a rigid body and is directed along the axis of rotation.

## 11.3 Conservation of Angular Momentum

- In the absence of external torques, a system's total angular momentum is conserved. This is the rotational counterpart to linear momentum being conserved when the external force on a system is zero.
- For a rigid body that changes its angular momentum in the absence of a net external torque, conservation of angular momentum gives  $I_f \omega_f = I_i \omega_i$ . This equation says that the angular velocity is inversely proportional to the moment of inertia. Thus, if the moment of inertia decreases, the angular velocity must increase to conserve angular momentum.
- Systems containing both point particles and rigid bodies can be analyzed using conservation of angular momentum. The angular momentum of all bodies in the system must be taken about a common axis.

## 11.4 Precession of a Gyroscope

- When a gyroscope is set on a pivot near the surface of Earth, it precesses around a vertical axis, since the torque is always horizontal and perpendicular to  $\vec{L}$ . If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque, and it rotates about a horizontal axis, falling over just as we would expect.
- The precessional angular velocity is given by  $\omega_p = \frac{rMg}{I\omega}$ , where  $r$  is the distance from the pivot to the center of mass of the gyroscope,  $I$  is the moment of inertia of the gyroscope's spinning disk,  $M$  is its mass, and  $\omega$  is the angular frequency of the gyroscope disk.

# CONCEPTUAL QUESTIONS

### 11.1 Rolling Motion

1. Can a round object released from rest at the top of a frictionless incline undergo rolling motion?
2. A cylindrical can of radius  $R$  is rolling across a horizontal surface without slipping. (a) After one complete revolution of the can, what is the distance that its center of mass has moved? (b) Would this distance be greater or smaller if slipping occurred?
3. A wheel is released from the top on an incline. Is the wheel most likely to slip if the incline is steep or gently sloped?

4. Which rolls down an inclined plane faster, a hollow cylinder or a solid sphere? Both have the same mass and radius.

5. A hollow sphere and a hollow cylinder of the same radius and mass roll up an incline without slipping and have the same initial center of mass velocity. Which object reaches a greater height before stopping?

### 11.2 Angular Momentum

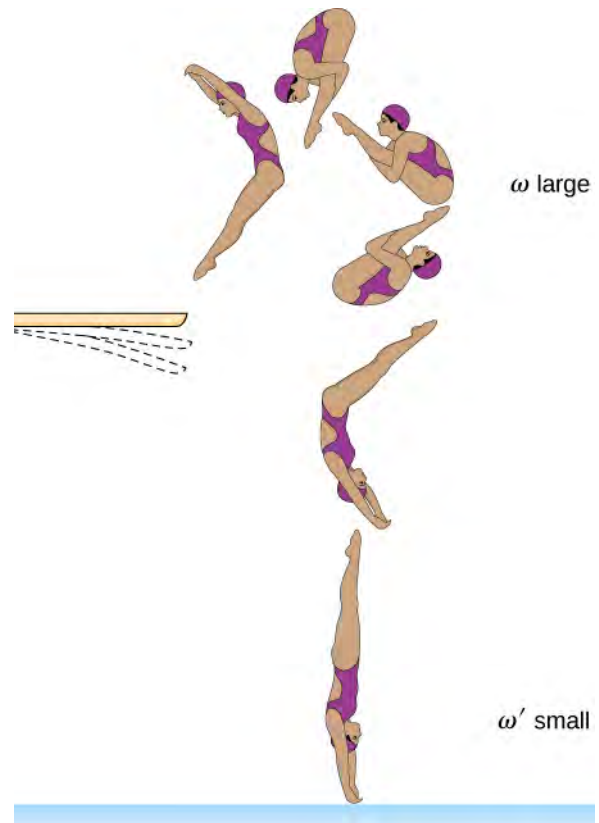
6. Can you assign an angular momentum to a particle without first defining a reference point?
7. For a particle traveling in a straight line, are there any

points about which the angular momentum is zero? Assume the line intersects the origin.

8. Under what conditions does a rigid body have angular momentum but not linear momentum?
9. If a particle is moving with respect to a chosen origin it has linear momentum. What conditions must exist for this particle's angular momentum to be zero about the chosen origin?
10. If you know the velocity of a particle, can you say anything about the particle's angular momentum?

### 11.3 Conservation of Angular Momentum

11. What is the purpose of the small propeller at the back of a helicopter that rotates in the plane perpendicular to the large propeller?
12. Suppose a child walks from the outer edge of a rotating merry-go-round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer. Assume the merry-go-round is spinning without friction.
13. As the rope of a tethered ball winds around a pole, what happens to the angular velocity of the ball?
14. Suppose the polar ice sheets broke free and floated toward Earth's equator without melting. What would happen to Earth's angular velocity?
15. Explain why stars spin faster when they collapse.
16. Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down (see below). Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momentum.



### 11.4 Precession of a Gyroscope

17. Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. When placed in the vehicle, they are put in a compartment that is separated from the main fuselage, such that changes in the orientation of the fuselage does not affect the orientation of the gyroscope. If the space vehicle is subjected to large forces and accelerations how can the direction of the gyroscopes angular momentum be constant at all times?
18. Earth precesses about its vertical axis with a period of 26,000 years. Discuss whether **Equation 11.12** can be used to calculate the precessional angular velocity of Earth.

## PROBLEMS

### 11.1 Rolling Motion

19. What is the angular velocity of a 75.0-cm-diameter tire on an automobile traveling at 90.0 km/h?
20. A boy rides his bicycle 2.00 km. The wheels have radius 30.0 cm. What is the total angle the tires rotate

through during his trip?

21. If the boy on the bicycle in the preceding problem accelerates from rest to a speed of 10.0 m/s in 10.0 s, what is the angular acceleration of the tires?
22. Formula One race cars have 66-cm-diameter tires. If a Formula One averages a speed of 300 km/h during a



race, what is the angular displacement in revolutions of the wheels if the race car maintains this speed for 1.5 hours?

23. A marble rolls down an incline at  $30^\circ$  from rest. (a) What is its acceleration? (b) How far does it go in 3.0 s?

24. Repeat the preceding problem replacing the marble with a solid cylinder. Explain the new result.

25. A rigid body with a cylindrical cross-section is released from the top of a  $30^\circ$  incline. It rolls 10.0 m to the bottom in 2.60 s. Find the moment of inertia of the body in terms of its mass  $m$  and radius  $r$ .

26. A yo-yo can be thought of a solid cylinder of mass  $m$  and radius  $r$  that has a light string wrapped around its circumference (see below). One end of the string is held fixed in space. If the cylinder falls as the string unwinds without slipping, what is the acceleration of the cylinder?



27. A solid cylinder of radius 10.0 cm rolls down an incline with slipping. The angle of the incline is  $30^\circ$ . The coefficient of kinetic friction on the surface is 0.400. What is the angular acceleration of the solid cylinder? What is the linear acceleration?

28. A bowling ball rolls up a ramp 0.5 m high without slipping to storage. It has an initial velocity of its center of mass of 3.0 m/s. (a) What is its velocity at the top of the ramp? (b) If the ramp is 1 m high does it make it to the top?

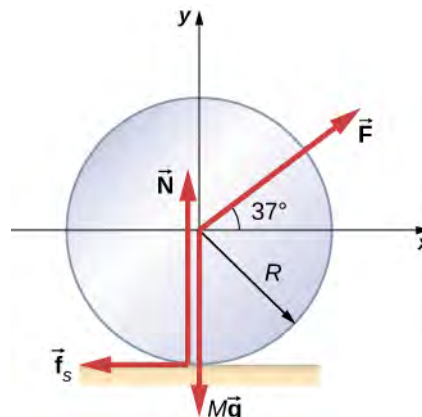
29. A 40.0-kg solid cylinder is rolling across a horizontal surface at a speed of 6.0 m/s. How much work is required to stop it?

30. A 40.0-kg solid sphere is rolling across a horizontal surface with a speed of 6.0 m/s. How much work is required to stop it? Compare results with the preceding problem.

31. A solid cylinder rolls up an incline at an angle of  $20^\circ$ . If it starts at the bottom with a speed of 10 m/s, how far up the incline does it travel?

32. A solid cylindrical wheel of mass  $M$  and radius  $R$  is pulled by a force  $\vec{F}$  applied to the center of the wheel at  $37^\circ$  to the horizontal (see the following figure). If the

wheel is to roll without slipping, what is the maximum value of  $|\vec{F}|$ ? The coefficients of static and kinetic friction are  $\mu_s = 0.40$  and  $\mu_k = 0.30$ .



33. A hollow cylinder is given a velocity of 5.0 m/s and rolls up an incline to a height of 1.0 m. If a hollow sphere of the same mass and radius is given the same initial velocity, how high does it roll up the incline?

## 11.2 Angular Momentum

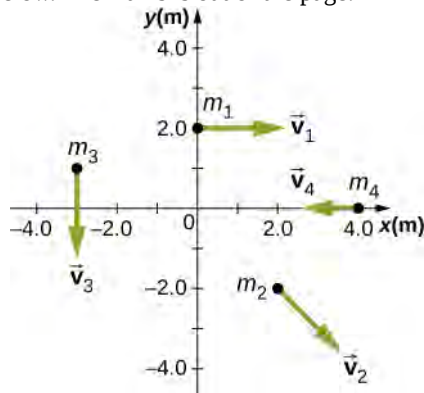
34. A 0.2-kg particle is travelling along the line  $y = 2.0$  m with a velocity 5.0 m/s. What is the angular momentum of the particle about the origin?

35. A bird flies overhead from where you stand at an altitude of 300.0 m and at a speed horizontal to the ground of 20.0 m/s. The bird has a mass of 2.0 kg. The radius vector to the bird makes an angle  $\theta$  with respect to the ground. The radius vector to the bird and its momentum vector lie in the  $xy$ -plane. What is the bird's angular momentum about the point where you are standing?

36. A Formula One race car with mass 750.0 kg is speeding through a course in Monaco and enters a circular turn at 220.0 km/h in the counterclockwise direction about the origin of the circle. At another part of the course, the car enters a second circular turn at 180 km/h also in the counterclockwise direction. If the radius of curvature of the first turn is 130.0 m and that of the second is 100.0 m, compare the angular momenta of the race car in each turn taken about the origin of the circular turn.

37. A particle of mass 5.0 kg has position vector  $\vec{r} = (2.0\hat{i} - 3.0\hat{j})\text{m}$  at a particular instant of time when its velocity is  $\vec{v} = (3.0\hat{i})\text{m/s}$  with respect to the origin. (a) What is the angular momentum of the particle? (b) If a force  $\vec{F} = 5.0\hat{j}\text{N}$  acts on the particle at this instant, what is the torque about the origin?

38. Use the right-hand rule to determine the directions of the angular momenta about the origin of the particles as shown below. The z-axis is out of the page.



39. Suppose the particles in the preceding problem have masses  $m_1 = 0.10 \text{ kg}$ ,  $m_2 = 0.20 \text{ kg}$ ,  $m_3 = 0.30 \text{ kg}$ ,  $m_4 = 0.40 \text{ kg}$ . The velocities of the particles are  $\vec{v}_1 = 2.0 \hat{i} \text{ m/s}$ ,  $\vec{v}_2 = (3.0 \hat{i} - 3.0 \hat{j}) \text{ m/s}$ ,  $\vec{v}_3 = -1.5 \hat{j} \text{ m/s}$ ,  $\vec{v}_4 = -4.0 \hat{i} \text{ m/s}$ . (a) Calculate the angular momentum of each particle about the origin. (b) What is the total angular momentum of the four-particle system about the origin?

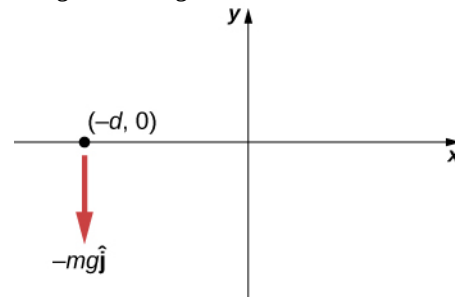
40. Two particles of equal mass travel with the same speed in opposite directions along parallel lines separated by a distance  $d$ . Show that the angular momentum of this two-particle system is the same no matter what point is used as the reference for calculating the angular momentum.

41. An airplane of mass  $4.0 \times 10^4 \text{ kg}$  flies horizontally at an altitude of 10 km with a constant speed of 250 m/s relative to Earth. (a) What is the magnitude of the airplane's angular momentum relative to a ground observer directly below the plane? (b) Does the angular momentum change as the airplane flies along its path?

42. At a particular instant, a 1.0-kg particle's position is  $\vec{r} = (2.0 \hat{i} - 4.0 \hat{j} + 6.0 \hat{k}) \text{ m}$ , its velocity is  $\vec{v} = (-1.0 \hat{i} + 4.0 \hat{j} + 1.0 \hat{k}) \text{ m/s}$ , and the force on it is  $\vec{F} = (10.0 \hat{i} + 15.0 \hat{j}) \text{ N}$ . (a) What is the angular momentum of the particle about the origin? (b) What is the torque on the particle about the origin? (c) What is the time rate of change of the particle's angular momentum at this instant?

43. A particle of mass  $m$  is dropped at the point  $(-d, 0)$  and falls vertically in Earth's gravitational field  $-g \hat{j}$ .

(a) What is the expression for the angular momentum of the particle around the z-axis, which points directly out of the page as shown below? (b) Calculate the torque on the particle around the z-axis. (c) Is the torque equal to the time rate of change of the angular momentum?



44. (a) Calculate the angular momentum of Earth in its orbit around the Sun. (b) Compare this angular momentum with the angular momentum of Earth about its axis.

45. A boulder of mass 20 kg and radius 20 cm rolls down a hill 15 m high from rest. What is its angular momentum when it is half way down the hill? (b) At the bottom?

46. A satellite is spinning at 6.0 rev/s. The satellite consists of a main body in the shape of a sphere of radius 2.0 m and mass 10,000 kg, and two antennas projecting out from the center of mass of the main body that can be approximated with rods of length 3.0 m each and mass 10 kg. The antenna's lie in the plane of rotation. What is the angular momentum of the satellite?

47. A propeller consists of two blades each 3.0 m in length and mass 120 kg each. The propeller can be approximated by a single rod rotating about its center of mass. The propeller starts from rest and rotates up to 1200 rpm in 30 seconds at a constant rate. (a) What is the angular momentum of the propeller at  $t = 10 \text{ s}$ ;  $t = 20 \text{ s}$ ? (b) What is the torque on the propeller?

48. A pulsar is a rapidly rotating neutron star. The Crab nebula pulsar in the constellation Taurus has a period of  $33.5 \times 10^{-3} \text{ s}$ , radius 10.0 km, and mass  $2.8 \times 10^{30} \text{ kg}$ . The pulsar's rotational period will increase over time due to the release of electromagnetic radiation, which doesn't change its radius but reduces its rotational energy. (a) What is the angular momentum of the pulsar? (b) Suppose the angular velocity decreases at a rate of  $10^{-14} \text{ rad/s}^2$ . What is the torque on the pulsar?

49. The blades of a wind turbine are 30 m in length and rotate at a maximum rotation rate of 20 rev/min. (a) If the blades are 6000 kg each and the rotor assembly has three blades, calculate the angular momentum of the turbine at this rotation rate. (b) What is the torque require to rotate the blades up to the maximum rotation rate in 5 minutes?

50. A roller coaster has mass 3000.0 kg and needs to make it safely through a vertical circular loop of radius 50.0 m. What is the minimum angular momentum of the coaster at the bottom of the loop to make it safely through? Neglect friction on the track. Take the coaster to be a point particle.

51. A mountain biker takes a jump in a race and goes airborne. The mountain bike is travelling at 10.0 m/s before it goes airborne. If the mass of the front wheel on the bike is 750 g and has radius 35 cm, what is the angular momentum of the spinning wheel in the air the moment the bike leaves the ground?

### 11.3 Conservation of Angular Momentum

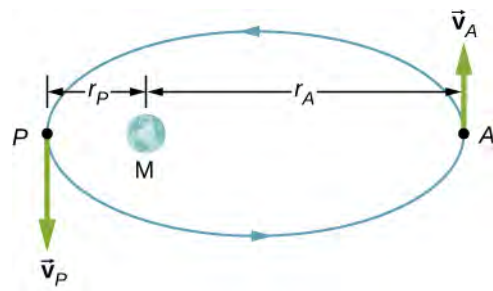
52. A disk of mass 2.0 kg and radius 60 cm with a small mass of 0.05 kg attached at the edge is rotating at 2.0 rev/s. The small mass suddenly separates from the disk. What is the disk's final rotation rate?

53. The Sun's mass is  $2.0 \times 10^{30}$  kg, its radius is  $7.0 \times 10^5$  km, and it has a rotational period of approximately 28 days. If the Sun should collapse into a white dwarf of radius  $3.5 \times 10^3$  km, what would its period be if no mass were ejected and a sphere of uniform density can model the Sun both before and after?

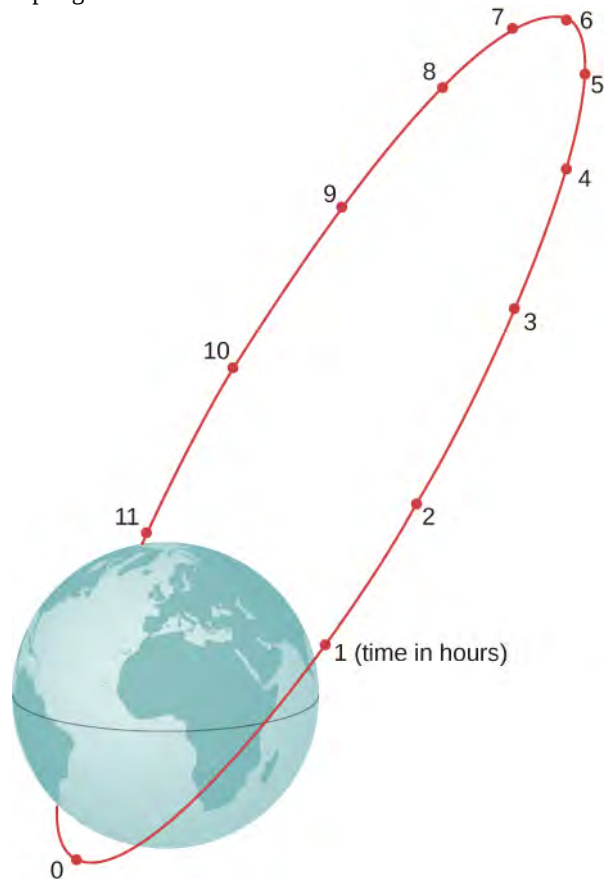
54. A cylinder with rotational inertia  $I_1 = 2.0 \text{ kg} \cdot \text{m}^2$  rotates clockwise about a vertical axis through its center with angular speed  $\omega_1 = 5.0 \text{ rad/s}$ . A second cylinder with rotational inertia  $I_2 = 1.0 \text{ kg} \cdot \text{m}^2$  rotates counterclockwise about the same axis with angular speed  $\omega_2 = 8.0 \text{ rad/s}$ . If the cylinders couple so they have the same rotational axis what is the angular speed of the combination? What percentage of the original kinetic energy is lost to friction?

55. A diver off the high board imparts an initial rotation with his body fully extended before going into a tuck and executing three back somersaults before hitting the water. If his moment of inertia before the tuck is  $16.9 \text{ kg} \cdot \text{m}^2$  and after the tuck during the somersaults is  $4.2 \text{ kg} \cdot \text{m}^2$ , what rotation rate must he impart to his body directly off the board and before the tuck if he takes 1.4 s to execute the somersaults before hitting the water?

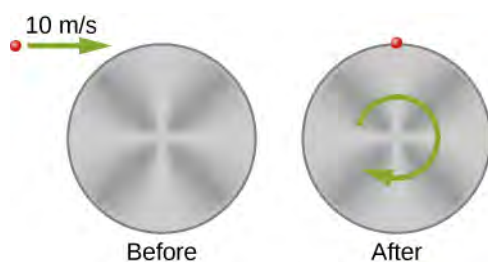
56. An Earth satellite has its apogee at 2500 km above the surface of Earth and perigee at 500 km above the surface of Earth. At apogee its speed is 730 m/s. What is its speed at perigee? Earth's radius is 6370 km (see below).



57. A Molniya orbit is a highly eccentric orbit of a communication satellite so as to provide continuous communications coverage for Scandinavian countries and adjacent Russia. The orbit is positioned so that these countries have the satellite in view for extended periods in time (see below). If a satellite in such an orbit has an apogee at 40,000.0 km as measured from the center of Earth and a velocity of 3.0 km/s, what would be its velocity at perigee measured at 200.0 km altitude?



58. Shown below is a small particle of mass 20 g that is moving at a speed of 10.0 m/s when it collides and sticks to the edge of a uniform solid cylinder. The cylinder is free to rotate about its axis through its center and is perpendicular to the page. The cylinder has a mass of 0.5 kg and a radius of 10 cm, and is initially at rest. (a) What is the angular velocity of the system after the collision? (b) How much kinetic energy is lost in the collision?



**59.** A bug of mass  $0.020\text{ kg}$  is at rest on the edge of a solid cylindrical disk ( $M = 0.10\text{ kg}$ ,  $R = 0.10\text{ m}$ ) rotating in a horizontal plane around the vertical axis through its center. The disk is rotating at  $10.0\text{ rad/s}$ . The bug crawls to the center of the disk. (a) What is the new angular velocity of the disk? (b) What is the change in the kinetic energy of the system? (c) If the bug crawls back to the outer edge of the disk, what is the angular velocity of the disk then? (d) What is the new kinetic energy of the system? (e) What is the cause of the increase and decrease of kinetic energy?

**60.** A uniform rod of mass  $200\text{ g}$  and length  $100\text{ cm}$  is free to rotate in a horizontal plane around a fixed vertical axis through its center, perpendicular to its length. Two small beads, each of mass  $20\text{ g}$ , are mounted in grooves along the rod. Initially, the two beads are held by catches on opposite sides of the rod's center,  $10\text{ cm}$  from the axis of rotation. With the beads in this position, the rod is rotating with an angular velocity of  $10.0\text{ rad/s}$ . When the catches are released, the beads slide outward along the rod. (a) What is the rod's angular velocity when the beads reach the ends of the rod? (b) What is the rod's angular velocity if the beads fly off the rod?

**61.** A merry-go-round has a radius of  $2.0\text{ m}$  and a moment of inertia  $300\text{ kg} \cdot \text{m}^2$ . A boy of mass  $50\text{ kg}$  runs tangent to the rim at a speed of  $4.0\text{ m/s}$  and jumps on. If the merry-go-round is initially at rest, what is the angular velocity after the boy jumps on?

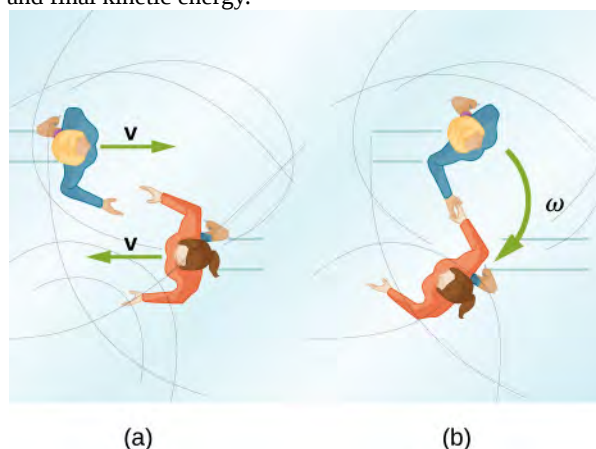
**62.** A playground merry-go-round has a mass of  $120\text{ kg}$  and a radius of  $1.80\text{ m}$  and it is rotating with an angular velocity of  $0.500\text{ rev/s}$ . What is its angular velocity after a  $22.0\text{-kg}$  child gets onto it by grabbing its outer edge? The child is initially at rest.

**63.** Three children are riding on the edge of a merry-go-round that is  $100\text{ kg}$ , has a  $1.60\text{-m}$  radius, and is spinning at  $20.0\text{ rpm}$ . The children have masses of  $22.0$ ,  $28.0$ , and  $33.0\text{ kg}$ . If the child who has a mass of  $28.0\text{ kg}$  moves to the center of the merry-go-round, what is the new angular velocity in rpm?

**64.** (a) Calculate the angular momentum of an ice skater spinning at  $6.00\text{ rev/s}$  given his moment of inertia is  $0.400\text{ kg} \cdot \text{m}^2$ . (b) He reduces his rate of spin (his angular

velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to  $1.25\text{ rev/s}$ . (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to  $3.00\text{ rev/s}$ . What average torque was exerted if this takes  $15.0\text{ s}$ ?

**65.** Twin skaters approach one another as shown below and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of  $2.50\text{ m/s}$  relative to the ice. Each has a mass of  $70.0\text{ kg}$ , and each has a center of mass located  $0.800\text{ m}$  from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.



**66.** A baseball catcher extends his arm straight up to catch a fast ball with a speed of  $40\text{ m/s}$ . The baseball is  $0.145\text{ kg}$  and the catcher's arm length is  $0.5\text{ m}$  and mass  $4.0\text{ kg}$ . (a) What is the angular velocity of the arm immediately after catching the ball as measured from the arm socket? (b) What is the torque applied if the catcher stops the rotation of his arm  $0.3\text{ s}$  after catching the ball?

**67.** In 2015, in Warsaw, Poland, Olivia Oliver of Nova Scotia broke the world record for being the fastest spinner on ice skates. She achieved a record  $342\text{ rev/min}$ , beating the existing Guinness World Record by  $34$  rotations. If an ice skater extends her arms at that rotation rate, what would be her new rotation rate? Assume she can be approximated by a  $45\text{-kg}$  rod that is  $1.7\text{ m}$  tall with a radius of  $15\text{ cm}$  in the record spin. With her arms stretched take the approximation of a rod of length  $130\text{ cm}$  with  $10\%$  of her body mass aligned perpendicular to the spin axis. Neglect frictional forces.

**68.** A satellite in a geosynchronous circular orbit is  $42,164.0\text{ km}$  from the center of Earth. A small asteroid collides with the satellite sending it into an elliptical orbit of apogee  $45,000.0\text{ km}$ . What is the speed of the satellite at apogee? Assume its angular momentum is conserved.

**69.** A gymnast does cartwheels along the floor and then



launches herself into the air and executes several flips in a tuck while she is airborne. If her moment of inertia when executing the cartwheels is  $13.5 \text{ kg} \cdot \text{m}^2$  and her spin rate is 0.5 rev/s, how many revolutions does she do in the air if her moment of inertia in the tuck is  $3.4 \text{ kg} \cdot \text{m}^2$  and she has 2.0 s to do the flips in the air?

**70.** The centrifuge at NASA Ames Research Center has a radius of 8.8 m and can produce forces on its payload of 20 *gs* or 20 times the force of gravity on Earth. (a) What is the angular momentum of a 20-kg payload that experiences 10 *gs* in the centrifuge? (b) If the driver motor was turned off in (a) and the payload lost 10 kg, what would be its new spin rate, taking into account there are no frictional forces present?

**71.** A ride at a carnival has four spokes to which pods are attached that can hold two people. The spokes are each 15 m long and are attached to a central axis. Each spoke has mass 200.0 kg, and the pods each have mass 100.0 kg. If the ride spins at 0.2 rev/s with each pod containing two 50.0-kg children, what is the new spin rate if all the children jump off the ride?

**72.** An ice skater is preparing for a jump with turns and has his arms extended. His moment of inertia is  $1.8 \text{ kg} \cdot \text{m}^2$  while his arms are extended, and he is spinning at 0.5 rev/s. If he launches himself into the air at 9.0 m/s at an angle of  $45^\circ$  with respect to the ice, how many revolutions can he execute while airborne if his moment of inertia in the air is  $0.5 \text{ kg} \cdot \text{m}^2$ ?

**73.** A space station consists of a giant rotating hollow cylinder of mass  $10^6 \text{ kg}$  including people on the station and a radius of 100.00 m. It is rotating in space at 3.30 rev/min in order to produce artificial gravity. If 100 people of an average mass of 65.00 kg spacewalk to an awaiting spaceship, what is the new rotation rate when all the people are off the station?

**74.** Neptune has a mass of  $1.0 \times 10^{26} \text{ kg}$  and is  $4.5 \times 10^9 \text{ km}$  from the Sun with an orbital period of 165 years. Planetesimals in the outer primordial solar system 4.5 billion years ago coalesced into Neptune over hundreds of millions of years. If the primordial disk that evolved into our present day solar system had a radius of  $10^{11}$

km and if the matter that made up these planetesimals that later became Neptune was spread out evenly on the edges of it, what was the orbital period of the outer edges of the primordial disk?

### 11.4 Precession of a Gyroscope

**75.** A gyroscope has a 0.5-kg disk that spins at 40 rev/s. The center of mass of the disk is 10 cm from a pivot which is also the radius of the disk. What is the precession angular velocity?

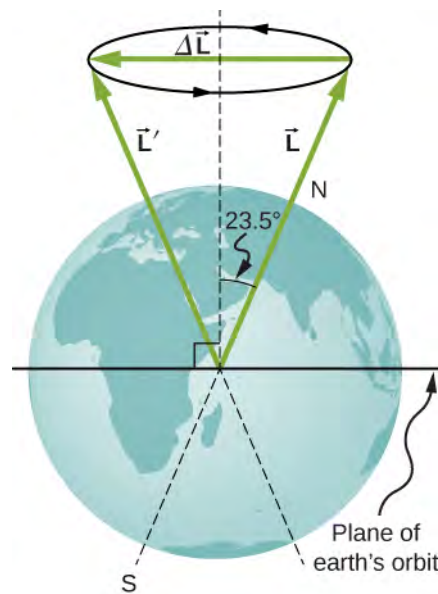
**76.** The precession angular velocity of a gyroscope is 1.0 rad/s. If the mass of the rotating disk is 0.4 kg and its radius is 30 cm, as well as the distance from the center of mass to the pivot, what is the rotation rate in rev/s of the disk?

**77.** The axis of Earth makes a  $23.5^\circ$  angle with a direction perpendicular to the plane of Earth's orbit. As shown below, this axis precesses, making one complete rotation in 25,780 y.

(a) Calculate the change in angular momentum in half this time.

(b) What is the average torque producing this change in angular momentum?

(c) If this torque were created by a pair of forces acting at the most effective point on the equator, what would the magnitude of each force be?



## ADDITIONAL PROBLEMS

**78.** A marble is rolling across the floor at a speed of 7.0 m/s when it starts up a plane inclined at  $30^\circ$  to the horizontal.

(a) How far along the plane does the marble travel before coming to a rest? (b) How much time elapses while the

marble moves up the plane?

**79.** Repeat the preceding problem replacing the marble with a hollow sphere. Explain the new results.

**80.** The mass of a hoop of radius 1.0 m is 6.0 kg. It rolls across a horizontal surface with a speed of 10.0 m/s. (a) How much work is required to stop the hoop? (b) If the hoop starts up a surface at  $30^\circ$  to the horizontal with a speed of 10.0 m/s, how far along the incline will it travel before stopping and rolling back down?

**81.** Repeat the preceding problem for a hollow sphere of the same radius and mass and initial speed. Explain the differences in the results.

**82.** A particle has mass 0.5 kg and is traveling along the line  $x = 5.0$  m at 2.0 m/s in the positive  $y$ -direction. What is the particle's angular momentum about the origin?

**83.** A 4.0-kg particle moves in a circle of radius 2.0 m. The angular momentum of the particle varies in time according to  $l = 5.0t^2$ . (a) What is the torque on the particle about the center of the circle at  $t = 3.4$  s? (b) What is the angular velocity of the particle at  $t = 3.4$  s?

**84.** A proton is accelerated in a cyclotron to  $5.0 \times 10^6$  m/s in 0.01 s. The proton follows a circular path. If the radius of the cyclotron is 0.5 km, (a) What is the angular momentum of the proton about the center at its maximum speed? (b) What is the torque on the proton about the center as it accelerates to maximum speed?

**85.** (a) What is the angular momentum of the Moon in its orbit around Earth? (b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.

**86.** A DVD is rotating at 500 rpm. What is the angular momentum of the DVD if has a radius of 6.0 cm and mass 20.0 g?

**87.** A potter's disk spins from rest up to 10 rev/s in 15 s. The disk has a mass 3.0 kg and radius 30.0 cm. What is the angular momentum of the disk at  $t = 5$  s,  $t = 10$  s?

**88.** Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What is the angular

momentum given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

**89.** A solid cylinder of mass 2.0 kg and radius 20 cm is rotating counterclockwise around a vertical axis through its center at 600 rev/min. A second solid cylinder of the same mass is rotating clockwise around the same vertical axis at 900 rev/min. If the cylinders couple so that they rotate about the same vertical axis, what is the angular velocity of the combination?

**90.** A boy stands at the center of a platform that is rotating without friction at 1.0 rev/s. The boy holds weights as far from his body as possible. At this position the total moment of inertia of the boy, platform, and weights is  $5.0 \text{ kg} \cdot \text{m}^2$ .

The boy draws the weights in close to his body, thereby decreasing the total moment of inertia to  $1.5 \text{ kg} \cdot \text{m}^2$ . (a) What is the final angular velocity of the platform? (b) By how much does the rotational kinetic energy increase?

**91.** Eight children, each of mass 40 kg, climb on a small merry-go-round. They position themselves evenly on the outer edge and join hands. The merry-go-round has a radius of 4.0 m and a moment of inertia  $1000.0 \text{ kg} \cdot \text{m}^2$ . After the merry-go-round is given an angular velocity of 6.0 rev/min, the children walk inward and stop when they are 0.75 m from the axis of rotation. What is the new angular velocity of the merry-go-round? Assume there is negligible frictional torque on the structure.

**92.** A thin meter stick of mass 150 g rotates around an axis perpendicular to the stick's long axis at an angular velocity of 240 rev/min. What is the angular momentum of the stick if the rotation axis (a) passes through the center of the stick? (b) Passes through one end of the stick?

**93.** A satellite in the shape of a sphere of mass 20,000 kg and radius 5.0 m is spinning about an axis through its center of mass. It has a rotation rate of 8.0 rev/s. Two antennas deploy in the plane of rotation extending from the center of mass of the satellite. Each antenna can be approximated as a rod has mass 200.0 kg and length 7.0 m. What is the new rotation rate of the satellite?

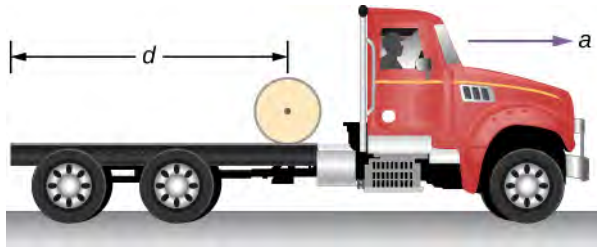
**94.** A top has moment of inertia  $3.2 \times 10^{-4} \text{ kg} \cdot \text{m}^2$  and radius 4.0 cm from the center of mass to the pivot point. If it spins at 20.0 rev/s and is precessing, how many revolutions does it precess in 10.0 s?

## CHALLENGE PROBLEMS

**95.** The truck shown below is initially at rest with solid

cylindrical roll of paper sitting on its bed. If the truck

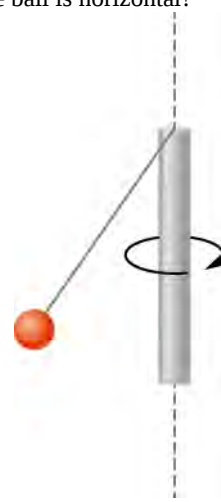
moves forward with a uniform acceleration  $a$ , what distance  $s$  does it move before the paper rolls off its back end? (*Hint: If the roll accelerates forward with  $a'$ , then it accelerates backward relative to the truck with an acceleration  $a - a'$ . Also,  $R\alpha = a - a'$ .)*



**96.** A bowling ball of radius 8.5 cm is tossed onto a bowling lane with speed 9.0 m/s. The direction of the toss is to the left, as viewed by the observer, so the bowling ball starts to rotate counterclockwise when in contact with the floor. The coefficient of kinetic friction on the lane is 0.3. (a) What is the time required for the ball to come to the point where it is not slipping? What is the distance  $d$  to the point where the ball is rolling without slipping?

**97.** A small ball of mass 0.50 kg is attached by a massless string to a vertical rod that is spinning as shown below. When the rod has an angular velocity of 6.0 rad/s, the string makes an angle of  $30^\circ$  with respect to the vertical. (a)

If the angular velocity is increased to 10.0 rad/s, what is the new angle of the string? (b) Calculate the initial and final angular momenta of the ball. (c) Can the rod spin fast enough so that the ball is horizontal?



**98.** A bug flying horizontally at 1.0 m/s collides and sticks to the end of a uniform stick hanging vertically. After the impact, the stick swings out to a maximum angle of  $5.0^\circ$  from the vertical before rotating back. If the mass of the stick is 10 times that of the bug, calculate the length of the stick.