7 | WORK AND KINETIC ENERGY

Figure 7.1 A sprinter exerts her maximum power to do as much work on herself as possible in the short time that her foot is in contact with the ground. This adds to her kinetic energy, preventing her from slowing down during the race. Pushing back hard on the track generates a reaction force that propels the sprinter forward to win at the finish. (credit: modification of work by Marie-Lan Nguyen)

Chapter Outline

Introduction

In this chapter, we discuss some basic physical concepts involved in every physical motion in the universe, going beyond the concepts of force and change in motion, which we discussed in **Motion in Two and Three Dimensions** and **Newton's Laws of Motion**. These concepts are work, kinetic energy, and power. We explain how these quantities are related to one another, which will lead us to a fundamental relationship called the work-energy theorem. In the next chapter, we generalize this idea to the broader principle of conservation of energy.

The application of Newton's laws usually requires solving differential equations that relate the forces acting on an object to the accelerations they produce. Often, an analytic solution is intractable or impossible, requiring lengthy numerical solutions or simulations to get approximate results. In such situations, more general relations, like the work-energy theorem (or the conservation of energy), can still provide useful answers to many questions and require a more modest amount of mathematical calculation. In particular, you will see how the work-energy theorem is useful in relating the speeds of a particle, at different points along its trajectory, to the forces acting on it, even when the trajectory is otherwise too complicated to deal with. Thus, some aspects of motion can be addressed with fewer equations and without vector decompositions.

7.1 | Work

Learning Objectives

By the end of this section, you will be able to:

- Represent the work done by any force
- Evaluate the work done for various forces

In physics, **work** represents a type of energy. Work is done when a force acts on something that undergoes a displacement from one position to another. Forces can vary as a function of position, and displacements can be along various paths between two points. We first define the increment of work dW done by a force $\vec{\mathbf{F}}$ acting through an infinitesimal displacement $d\vec{\mathbf{r}}$ as the dot product of these two vectors:

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = |\vec{\mathbf{F}}||d\vec{\mathbf{r}}|\cos\theta. \tag{7.1}$$

Then, we can add up the contributions for infinitesimal displacements, along a path between two positions, to get the total work.

Work Done by a Force

The work done by a force is the integral of the force with respect to displacement along the path of the displacement:

$$W_{AB} = \int_{\text{path } AB} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}. \tag{7.2}$$

The vectors involved in the definition of the work done by a force acting on a particle are illustrated in **Figure 7.2**.

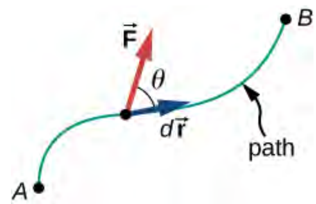


Figure 7.2 Vectors used to define work. The force acting on a particle and its infinitesimal displacement are shown at one point along the path between A and B. The infinitesimal work is the dot product of these two vectors; the total work is the integral of the dot product along the path.

We choose to express the dot product in terms of the magnitudes of the vectors and the cosine of the angle between them, because the meaning of the dot product for work can be put into words more directly in terms of magnitudes and angles. We could equally well have expressed the dot product in terms of the various components introduced in **Vectors**. In two dimensions, these were the x- and y-components in Cartesian coordinates, or the r- and φ-components in polar coordinates; in three dimensions, it was just x-, y-, and z-components. Which choice is more convenient depends on the situation. In words, you can express **Equation 7.1** for the work done by a force acting over a displacement as a product of one component acting parallel to the other component. From the properties of vectors, it doesn't matter if you take the component of the force parallel to the displacement or the component of the displacement parallel to the force—you get the same result either way.

Recall that the magnitude of a force times the cosine of the angle the force makes with a given direction is the component of the force in the given direction. The components of a vector can be positive, negative, or zero, depending on whether the angle between the vector and the component-direction is between $0°$ and $90°$ or $90°$ and $180°$, or is equal to $90°$. As a result, the work done by a force can be positive, negative, or zero, depending on whether the force is generally in the direction of the displacement, generally opposite to the displacement, or perpendicular to the displacement. The maximum work is done by a given force when it is along the direction of the displacement ($\cos\theta = \pm 1$), and zero work is done when the force is perpendicular to the displacement ($\cos\theta = 0$).

The units of work are units of force multiplied by units of length, which in the SI system is newtons times meters, $\text{N} \cdot \text{m}$. This combination is called a joule, for historical reasons that we will mention later, and is abbreviated as J. In the English system, still used in the United States, the unit of force is the pound (lb) and the unit of distance is the foot (ft), so the unit of work is the foot-pound $(\text{ft} \cdot \text{lb})$.

Work Done by Constant Forces and Contact Forces

The simplest work to evaluate is that done by a force that is constant in magnitude and direction. In this case, we can factor out the force; the remaining integral is just the total displacement, which only depends on the end points *A* and *B*, but not on the path between them:

$$W_{AB} = \vec{\mathbf{F}} \cdot \int_A^B d\vec{\mathbf{r}} = \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A) = |\vec{\mathbf{F}}||\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A| \cos\theta \quad \text{(constant force)}.$$

We can also see this by writing out **Equation 7.2** in Cartesian coordinates and using the fact that the components of the force are constant:

$$\begin{aligned} W_{AB} &= \int_{\text{path } AB} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path } AB} (F_x dx + F_y dy + F_z dz) = F_x \int_A^B dx + F_y \int_A^B dy + F_z \int_A^B dz \\ &= F_x (x_B - x_A) + F_y (y_B - y_A) + F_z (z_B - z_A) = \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A). \end{aligned}$$

Figure 7.3(a) shows a person exerting a constant force $\vec{\mathbf{F}}$ along the handle of a lawn mower, which makes an angle θ with the horizontal. The horizontal displacement of the lawn mower, over which the force acts, is $\vec{\mathbf{d}}$. The work done on the lawn mower is $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = Fd\cos\theta$, which the figure also illustrates as the horizontal component of the force times the magnitude of the displacement.

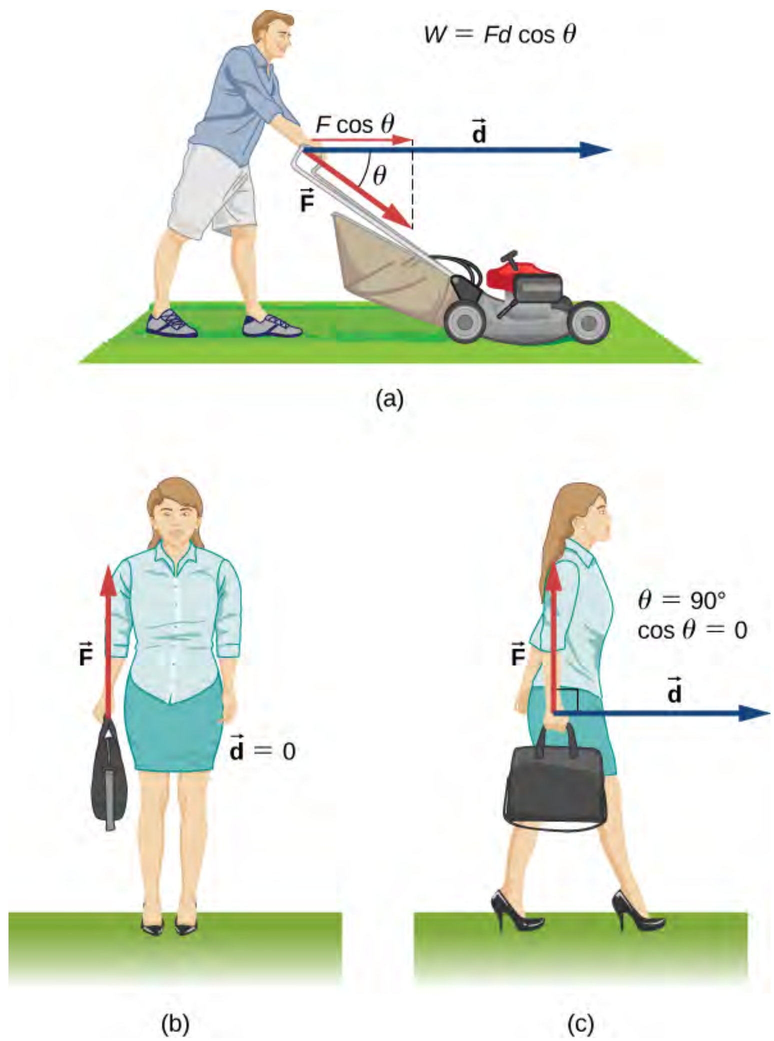


Figure 7.3 Work done by a constant force. (a) A person pushes a lawn mower with a constant force. The component of the force parallel to the displacement is the work done, as shown in the equation in the figure. (b) A person holds a briefcase. No work is done because the displacement is zero. (c) The person in (b) walks horizontally while holding the briefcase. No work is done because $\cos\theta$ is zero.

Figure 7.3(b) shows a person holding a briefcase. The person must exert an upward force, equal in magnitude to the weight of the briefcase, but this force does no work, because the displacement over which it acts is zero. So why do you eventually feel tired just holding the briefcase, if you're not doing any work on it? The answer is that muscle fibers in your arm are contracting and doing work inside your arm, even though the force your muscles exert externally on the briefcase doesn't do any work on it. (Part of the force you exert could also be tension in the bones and ligaments of your arm, but other muscles in your body would be doing work to maintain the position of your arm.)

In **Figure 7.3**(c), where the person in (b) is walking horizontally with constant speed, the work done by the person on the briefcase is still zero, but now because the angle between the force exerted and the displacement is $90°$ ($\vec{\mathbf{F}}$ perpendicular to $\vec{\mathbf{d}}$) and $\cos 90° = 0$.

Example 7.1

Calculating the Work You Do to Push a Lawn Mower

How much work is done on the lawn mower by the person in **Figure 7.3**(a) if he exerts a constant force of 75.0

N at an angle $35°$ below the horizontal and pushes the mower 25.0 m on level ground?

Strategy

We can solve this problem by substituting the given values into the definition of work done on an object by a constant force, stated in the equation $W = Fd\cos\theta$. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$$W = Fd\cos\theta.$$

Substituting the known values gives

$$W = (75.0\text{ N})(25.0\text{ m})\cos(35.0°) = 1.54 \times 10^3\text{ J}.$$

Significance

Even though one and a half kilojoules may seem like a lot of work, we will see in **Potential Energy and Conservation of Energy** that it's only about as much work as you could do by burning one sixth of a gram of fat.

When you mow the grass, other forces act on the lawn mower besides the force you exert—namely, the contact force of the ground and the gravitational force of Earth. Let's consider the work done by these forces in general. For an object moving on a surface, the displacement $d\vec{\mathbf{r}}$ is tangent to the surface. The part of the contact force on the object that is perpendicular to the surface is the normal force $\vec{\mathbf{N}}$. Since the cosine of the angle between the normal and the tangent to a surface is zero, we have

$$dW_{\text{N}} = \vec{\mathbf{N}} \cdot d\vec{\mathbf{r}} = \vec{\mathbf{0}}.$$

The normal force never does work under these circumstances. (Note that if the displacement $d\vec{\mathbf{r}}$ did have a relative component perpendicular to the surface, the object would either leave the surface or break through it, and there would no longer be any normal contact force. However, if the object is more than a particle, and has an internal structure, the normal contact force can do work on it, for example, by displacing it or deforming its shape. This will be mentioned in the next chapter.)

The part of the contact force on the object that is parallel to the surface is friction, $\vec{\mathbf{f}}$. For this object sliding along the surface, kinetic friction $\vec{\mathbf{f}}_{\text{k}}$ is opposite to $d\vec{\mathbf{r}}$, relative to the surface, so the work done by kinetic friction is negative. If the magnitude of $\vec{\mathbf{f}}_{\text{k}}$ is constant (as it would be if all the other forces on the object were constant), then the work done by friction is

$$W_{\text{fr}} = \int_A^B \vec{\mathbf{f}}_k \cdot d\vec{\mathbf{r}} = -f_k \int_A^B |dr| = -f_k |l_{AB}|, \tag{7.3}$$

where $|l_{AB}|$ is the path length on the surface. (Note that, especially if the work done *by* a force is negative, people may refer to the work done *against* this force, where $dW_{\text{against}} = -dW_{\text{by}}$. The work done against a force may also be viewed as the work required to overcome this force, as in "How much work is required to overcome…?") The force of static friction, however, can do positive or negative work. When you walk, the force of static friction exerted by the ground on your back foot accelerates you for part of each step. If you're slowing down, the force of the ground on your front foot decelerates you. If you're driving your car at the speed limit on a straight, level stretch of highway, the negative work done by kinetic friction of air resistance is balanced by the positive work done by the static friction of the road on the drive wheels. You can pull the rug out from under an object in such a way that it slides backward relative to the rug, but forward relative to the floor. In this case, kinetic friction exerted by the rug on the object could be in the same direction as the displacement of the object, relative to the floor, and do positive work. The bottom line is that you need to analyze each particular case to determine the work done by the forces, whether positive, negative or zero.

Example 7.2

Moving a Couch

You decide to move your couch to a new position on your horizontal living room floor. The normal force on the couch is 1 kN and the coefficient of friction is 0.6. (a) You first push the couch 3 m parallel to a wall and then 1 m perpendicular to the wall (*A* to *B* in **Figure 7.4**). How much work is done by the frictional force? (b) You don't like the new position, so you move the couch straight back to its original position (*B* to *A* in **Figure 7.4**). What was the total work done against friction moving the couch away from its original position and back again?

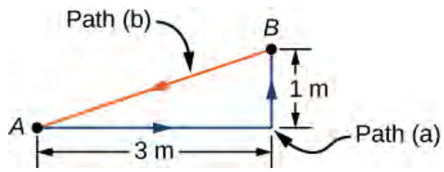


Figure 7.4 Top view of paths for moving a couch.

Strategy

The magnitude of the force of kinetic friction on the couch is constant, equal to the coefficient of friction times the normal force, $f_K = \mu_K N$. Therefore, the work done by it is $W_{\text{fr}} = -f_K d$, where *d* is the path length traversed. The segments of the paths are the sides of a right triangle, so the path lengths are easily calculated. In part (b), you can use the fact that the work done against a force is the negative of the work done by the force.

Solution

a. The work done by friction is

$$W = -(0.6)(1\text{ kN})(3\text{ m} + 1\text{ m}) = -2.4\text{ kJ}.$$

b. The length of the path along the hypotenuse is $\sqrt{10}\text{ m}$, so the total work done against friction is

$$W = (0.6)(1\text{ kN})(3\text{ m} + 1\text{ m} + \sqrt{10}\text{ m}) = 4.3\text{ kJ}.$$

Significance

The total path over which the work of friction was evaluated began and ended at the same point (it was a closed path), so that the total displacement of the couch was zero. However, the total work was not zero. The reason is that forces like friction are classified as nonconservative forces, or dissipative forces, as we discuss in the next chapter.

7.1 Check Your Understanding Can kinetic friction ever be a constant force for all paths?

The other force on the lawn mower mentioned above was Earth's gravitational force, or the weight of the mower. Near the surface of Earth, the gravitational force on an object of mass *m* has a constant magnitude, *mg*, and constant direction, vertically down. Therefore, the work done by gravity on an object is the dot product of its weight and its displacement. In many cases, it is convenient to express the dot product for gravitational work in terms of the *x*-, *y*-, and *z*-components of the vectors. A typical coordinate system has the *x*-axis horizontal and the *y*-axis vertically up. Then the gravitational force is $-mg\,\hat{\mathbf{j}}$, so the work done by gravity, over any path from *A* to *B*, is

$$W_{\text{grav},\,AB} = -mg\,\hat{\mathbf{j}} \cdot (\vec{\mathbf{r}}_B - \vec{\mathbf{r}}_A) = -mg(y_B - y_A). \tag{7.4}$$

The work done by a constant force of gravity on an object depends only on the object's weight and the difference in height through which the object is displaced. Gravity does negative work on an object that moves upward ($y_B > y_A$), or, in other words, you must do positive work against gravity to lift an object upward. Alternately, gravity does positive work on an object that moves downward ($y_B < y_A$), or you do negative work against gravity to "lift" an object downward, controlling its descent so it doesn't drop to the ground. ("Lift" is used as opposed to "drop".)

Example 7.3

Shelving a Book

You lift an oversized library book, weighing 20 N, 1 m vertically down from a shelf, and carry it 3 m horizontally to a table (**Figure 7.5**). How much work does gravity do on the book? (b) When you're finished, you move the book in a straight line back to its original place on the shelf. What was the total work done against gravity, moving the book away from its original position on the shelf and back again?

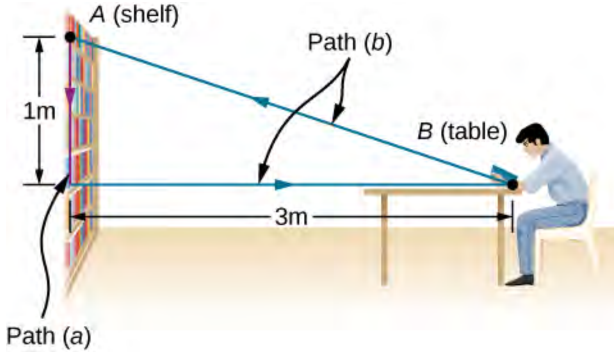


Figure 7.5 Side view of the paths for moving a book to and from a shelf.

Strategy

We have just seen that the work done by a constant force of gravity depends only on the weight of the object moved and the difference in height for the path taken, $W_{AB} = -mg(y_B - y_A)$. We can evaluate the difference in height to answer (a) and (b).

Solution

a. Since the book starts on the shelf and is lifted down $y_B - y_A = -1\text{ m}$, we have

$$W = -(20\text{ N})(-1\text{ m}) = 20\text{ J}.$$

b. There is zero difference in height for any path that begins and ends at the same place on the shelf, so $W = 0$.

Significance

Gravity does positive work (20 J) when the book moves down from the shelf. The gravitational force between two objects is an attractive force, which does positive work when the objects get closer together. Gravity does zero work (0 J) when the book moves horizontally from the shelf to the table and negative work (−20 J) when the book moves from the table back to the shelf. The total work done by gravity is zero $[20\text{ J} + 0\text{ J} + (-20\text{ J}) = 0]$.

Unlike friction or other dissipative forces, described in **Example 7.2**, the total work done against gravity, over any closed path, is zero. Positive work is done against gravity on the upward parts of a closed path, but an equal amount of negative work is done against gravity on the downward parts. In other words, work done *against* gravity, lifting an object *up*, is "given back" when the object comes back down. Forces like gravity (those that do zero work over any closed path) are classified as conservative forces and play an important role in physics.

7.2 Check Your Understanding Can Earth's gravity ever be a constant force for all paths?

Work Done by Forces that Vary

In general, forces may vary in magnitude and direction at points in space, and paths between two points may be curved. The infinitesimal work done by a variable force can be expressed in terms of the components of the force and the displacement along the path,

$$dW = F_x dx + F_y dy + F_z dz.$$

Here, the components of the force are functions of position along the path, and the displacements depend on the equations of the path. (Although we chose to illustrate dW in Cartesian coordinates, other coordinates are better suited to some situations.) **Equation 7.2** defines the total work as a line integral, or the limit of a sum of infinitesimal amounts of work. The physical concept of work is straightforward: you calculate the work for tiny displacements and add them up. Sometimes the mathematics can seem complicated, but the following example demonstrates how cleanly they can operate.

Example 7.4

Work Done by a Variable Force over a Curved Path

An object moves along a parabolic path $y = (0.5\text{ m}^{-1})x^2$ from the origin $A = (0, 0)$ to the point $B = (2\text{ m}, 2\text{ m})$ under the action of a force $\vec{\mathbf{F}} = (5\text{ N/m})y\hat{\mathbf{i}} + (10\text{ N/m})x\hat{\mathbf{j}}$ (**Figure 7.6**). Calculate the work done.

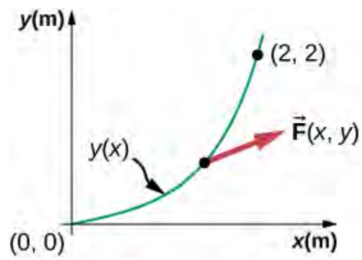


Figure 7.6 The parabolic path of a particle acted on by a given force.

Strategy

The components of the force are given functions of x and y. We can use the equation of the path to express y and dy in terms of x and dx; namely,

$$y = (0.5\text{ m}^{-1})x^2 \text{ and } dy = 2(0.5\text{ m}^{-1})x dx.$$

Then, the integral for the work is just a definite integral of a function of x.

Solution

The infinitesimal element of work is

$$\begin{aligned} dW &= F_x dx + F_y dy = (5\text{ N/m})y dx + (10\text{ N/m})x dy \\ &= (5\text{ N/m})(0.5\text{ m}^{-1})x^2 dx + (10\text{ N/m})2(0.5\text{ m}^{-1})x^2 dx = (12.5\text{ N/m}^2)x^2 dx. \end{aligned}$$

The integral of x^2 is $x^3/3$, so

$$W = \int_0^{2\text{ m}} (12.5\text{ N/m}^2)x^2 dx = (12.5\text{ N/m}^2)\frac{x^3}{3}\Big|_0^{2\text{ m}} = (12.5\text{ N/m}^2)\left(\frac{8}{3}\right) = 33.3\text{ J}.$$

Significance

This integral was not hard to do. You can follow the same steps, as in this example, to calculate line integrals representing work for more complicated forces and paths. In this example, everything was given in terms of x- and y-components, which are easiest to use in evaluating the work in this case. In other situations, magnitudes and angles might be easier.

7.3 Check Your Understanding Find the work done by the same force in **Example 7.4** over a cubic path, $y = (0.25\ \text{m}^{-2})x^3$, between the same points $A = (0,\ 0)$ and $B = (2\ \text{m}, 2\ \text{m})$.

You saw in **Example 7.4** that to evaluate a line integral, you could reduce it to an integral over a single variable or parameter. Usually, there are several ways to do this, which may be more or less convenient, depending on the particular case. In **Example 7.4**, we reduced the line integral to an integral over x, but we could equally well have chosen to reduce everything to a function of y. We didn't do that because the functions in y involve the square root and fractional exponents, which may be less familiar, but for illustrative purposes, we do this now. Solving for x and dx, in terms of y, along the parabolic path, we get

$$x = \sqrt{y/(0.5\ \text{m}^{-1})} = \sqrt{(2\ \text{m})y} \text{ and } dx = \sqrt{(2\ \text{m})} \times \tfrac{1}{2}dy/\sqrt{y} = dy/\sqrt{(2\ \text{m}^{-1})y}.$$

The components of the force, in terms of y, are

$$F_x = (5\ \text{N/m})y \text{ and } F_y = (10\ \text{N/m})x = (10\ \text{N/m})\sqrt{(2\ \text{m})y},$$

so the infinitesimal work element becomes

$$\begin{aligned} dW &= F_x dx + F_y dy = \frac{(5\ \text{N/m})y\,dy}{\sqrt{(2\ \text{m}^{-1})y}} + (10\ \text{N/m})\sqrt{(2\ \text{m})y}\,dy \\ &= (5\ \text{N}\cdot\text{m}^{-1/2})\left(\frac{1}{\sqrt{2}} + 2\sqrt{2}\right)\sqrt{y}\,dy = (17.7\ \text{N}\cdot\text{m}^{-1/2})y^{1/2}\,dy. \end{aligned}$$

The integral of $y^{1/2}$ is $\frac{2}{3}y^{3/2}$, so the work done from A to B is

$$W = \int_0^{2\ \text{m}} (17.7\ \text{N}\cdot\text{m}^{-1/2})y^{1/2}\,dy = (17.7\ \text{N}\cdot\text{m}^{-1/2})\tfrac{2}{3}(2\ \text{m})^{3/2} = 33.3\ \text{J}.$$

As expected, this is exactly the same result as before.

One very important and widely applicable variable force is the force exerted by a perfectly elastic spring, which satisfies Hooke's law $\vec{\mathbf{F}} = -k\Delta\vec{\mathbf{x}}$, where k is the spring constant, and $\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_{\text{eq}}$ is the displacement from the spring's unstretched (equilibrium) position (**Newton's Laws of Motion**). Note that the unstretched position is only the same as the equilibrium position if no other forces are acting (or, if they are, they cancel one another). Forces between molecules, or in any system undergoing small displacements from a stable equilibrium, behave approximately like a spring force.

To calculate the work done by a spring force, we can choose the x-axis along the length of the spring, in the direction of increasing length, as in **Figure 7.7**, with the origin at the equilibrium position $x_{\text{eq}} = 0$. (Then positive x corresponds to a stretch and negative x to a compression.) With this choice of coordinates, the spring force has only an x-component, $F_x = -kx$, and the work done when x changes from x_A to x_B is

$$W_{\text{spring},\,AB} = \int_A^B F_x dx = -k\int_A^B x dx = -k\frac{x^2}{2}\Big|_A^B = -\frac{1}{2}k\left(x_B^2 - x_A^2\right). \tag{7.5}$$

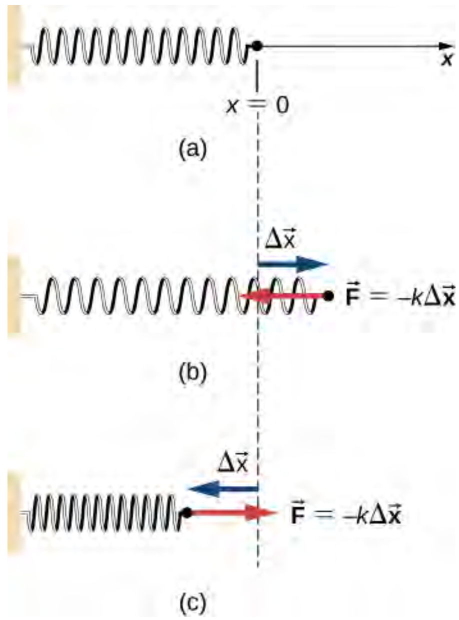


Figure 7.7 (a) The spring exerts no force at its equilibrium position. The spring exerts a force in the opposite direction to (b) an extension or stretch, and (c) a compression.

Notice that W_{AB} depends only on the starting and ending points, *A* and *B*, and is independent of the actual path between them, as long as it starts at *A* and ends at *B*. That is, the actual path could involve going back and forth before ending.

Another interesting thing to notice about **Equation 7.5** is that, for this one-dimensional case, you can readily see the correspondence between the work done by a force and the area under the curve of the force versus its displacement. Recall that, in general, a one-dimensional integral is the limit of the sum of infinitesimals, $f(x)dx$, representing the area of strips, as shown in **Figure 7.8**. In **Equation 7.5**, since $F = -kx$ is a straight line with slope $-k$, when plotted versus *x*, the "area" under the line is just an algebraic combination of triangular "areas," where "areas" above the *x*-axis are positive and those below are negative, as shown in **Figure 7.9**. The magnitude of one of these "areas" is just one-half the triangle's base, along the *x*-axis, times the triangle's height, along the force axis. (There are quotation marks around "area" because this base-height product has the units of work, rather than square meters.)

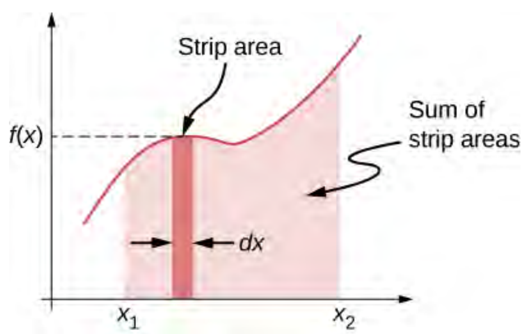


Figure 7.8 A curve of *f(x)* versus *x* showing the area of an infinitesimal strip, *f(x)dx*, and the sum of such areas, which is the integral of *f(x)* from x_1 to x_2.

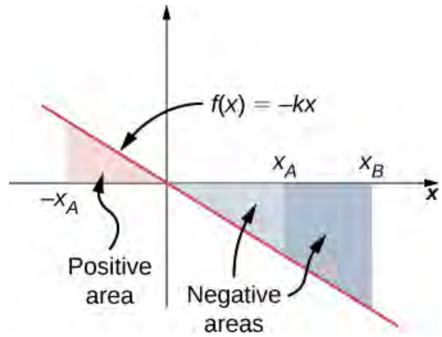


Figure 7.9 Curve of the spring force $f(x) = -kx$ versus x, showing areas under the line, between x_A and x_B, for both positive and negative values of x_A. When x_A is negative, the total area under the curve for the integral in **Equation 7.5** is the sum of positive and negative triangular areas. When x_A is positive, the total area under the curve is the difference between two negative triangles.

Example 7.5

Work Done by a Spring Force

A perfectly elastic spring requires 0.54 J of work to stretch 6 cm from its equilibrium position, as in **Figure 7.7**(b). (a) What is its spring constant k? (b) How much work is required to stretch it an additional 6 cm?

Strategy

Work "required" means work done against the spring force, which is the negative of the work in **Equation 7.5**, that is

$$W = \frac{1}{2}k(x_B^2 - x_A^2).$$

For part (a), $x_A = 0$ and $x_B = 6\text{cm}$; for part (b), $x_B = 6\text{cm}$ and $x_B = 12\text{cm}$. In part (a), the work is given and you can solve for the spring constant; in part (b), you can use the value of k, from part (a), to solve for the work.

Solution

a. $W = 0.54\text{ J} = \frac{1}{2}k[(6\text{ cm})^2 - 0]$, so $k = 3\text{ N/cm}$.

b. $W = \frac{1}{2}(3\text{ N/cm})[(12\text{ cm})^2 - (6\text{ cm})^2] = 1.62\text{ J}$.

Significance

Since the work done by a spring force is independent of the path, you only needed to calculate the difference in the quantity $½kx^2$ at the end points. Notice that the work required to stretch the spring from 0 to 12 cm is four times that required to stretch it from 0 to 6 cm, because that work depends on the square of the amount of stretch from equilibrium, $½kx^2$. In this circumstance, the work to stretch the spring from 0 to 12 cm is also equal to the work for a composite path from 0 to 6 cm followed by an additional stretch from 6 cm to 12 cm. Therefore, $4W(0\text{ cm to }6\text{ cm}) = W(0\text{ cm to }6\text{ cm}) + W(6\text{ cm to }12\text{ cm})$, or $W(6\text{ cm to }12\text{ cm}) = 3W(0\text{ cm to }6\text{ cm})$, as we found above.

7.4 Check Your Understanding The spring in **Example 7.5** is compressed 6 cm from its equilibrium length. (a) Does the spring force do positive or negative work and (b) what is the magnitude?

7.2 | Kinetic Energy

Learning Objectives

By the end of this section, you will be able to:

- Calculate the kinetic energy of a particle given its mass and its velocity or momentum
- Evaluate the kinetic energy of a body, relative to different frames of reference

It's plausible to suppose that the greater the velocity of a body, the greater effect it could have on other bodies. This does not depend on the direction of the velocity, only its magnitude. At the end of the seventeenth century, a quantity was introduced into mechanics to explain collisions between two perfectly elastic bodies, in which one body makes a head-on collision with an identical body at rest. The first body stops, and the second body moves off with the initial velocity of the first body. (If you have ever played billiards or croquet, or seen a model of Newton's Cradle, you have observed this type of collision.) The idea behind this quantity was related to the forces acting on a body and was referred to as "the energy of motion." Later on, during the eighteenth century, the name **kinetic energy** was given to energy of motion.

With this history in mind, we can now state the classical definition of kinetic energy. Note that when we say "classical," we mean non-relativistic, that is, at speeds much less that the speed of light. At speeds comparable to the speed of light, the special theory of relativity requires a different expression for the kinetic energy of a particle, as discussed in **Relativity (http://cnx.org/content/m58555/latest/)** .

Since objects (or systems) of interest vary in complexity, we first define the kinetic energy of a particle with mass *m*.

Kinetic Energy

The kinetic energy of a particle is one-half the product of the particle's mass *m* and the square of its speed *v*:

$$K = \frac{1}{2}mv^2. \tag{7.6}$$

We then extend this definition to any system of particles by adding up the kinetic energies of all the constituent particles:

$$K = \sum \frac{1}{2}mv^2. \tag{7.7}$$

Note that just as we can express Newton's second law in terms of either the rate of change of momentum or mass times the rate of change of velocity, so the kinetic energy of a particle can be expressed in terms of its mass and momentum $(\vec{\mathbf{p}} = m\vec{\mathbf{v}})$, instead of its mass and velocity. Since $v = p/m$, we see that

$$K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}$$

also expresses the kinetic energy of a single particle. Sometimes, this expression is more convenient to use than **Equation 7.6**.

The units of kinetic energy are mass times the square of speed, or $\text{kg} \cdot \text{m}^2/\text{s}^2$. But the units of force are mass times acceleration, $\text{kg} \cdot \text{m/s}^2$, so the units of kinetic energy are also the units of force times distance, which are the units of work, or joules. You will see in the next section that work and kinetic energy have the same units, because they are different forms of the same, more general, physical property.

Example 7.6

Kinetic Energy of an Object

(a) What is the kinetic energy of an 80-kg athlete, running at 10 m/s? (b) The Chicxulub crater in Yucatan, one of the largest existing impact craters on Earth, is thought to have been created by an asteroid, traveling at 22 km/s and releasing 4.2×10^{23} J of kinetic energy upon impact. What was its mass? (c) In nuclear reactors, thermal neutrons, traveling at about 2.2 km/s, play an important role. What is the kinetic energy of such a particle?

Strategy

To answer these questions, you can use the definition of kinetic energy in **Equation 7.6**. You also have to look up the mass of a neutron.

Solution

Don’t forget to convert km into m to do these calculations, although, to save space, we omitted showing these conversions.

a. $K = \frac{1}{2}(80\,\text{kg})(10\,\text{m/s})^2 = 4.0\,\text{kJ}.$

b. $m = 2K/v^2 = 2(4.2 \times 10^{23}\,\text{J})/(22\,\text{km/s})^2 = 1.7 \times 10^{15}\ \text{kg}.$

c. $K = \frac{1}{2}(1.68 \times 10^{-27}\ \text{kg})(2.2\,\text{km/s})^2 = 4.1 \times 10^{-21}\ \text{J}.$

Significance

In this example, we used the way mass and speed are related to kinetic energy, and we encountered a very wide range of values for the kinetic energies. Different units are commonly used for such very large and very small values. The energy of the impactor in part (b) can be compared to the explosive yield of TNT and nuclear explosions, $1\ \text{megaton} = 4.18 \times 10^{15}\ \text{J}$. The Chicxulub asteroid’s kinetic energy was about a hundred million megatons. At the other extreme, the energy of subatomic particle is expressed in electron-volts, $1\ \text{eV} = 1.6 \times 10^{-19}\ \text{J}$. The thermal neutron in part (c) has a kinetic energy of about one fortieth of an electron-volt.

7.5 Check Your Understanding (a) A car and a truck are each moving with the same kinetic energy. Assume that the truck has more mass than the car. Which has the greater speed? (b) A car and a truck are each moving with the same speed. Which has the greater kinetic energy?

Because velocity is a relative quantity, you can see that the value of kinetic energy must depend on your frame of reference. You can generally choose a frame of reference that is suited to the purpose of your analysis and that simplifies your calculations. One such frame of reference is the one in which the observations of the system are made (likely an external frame). Another choice is a frame that is attached to, or moves with, the system (likely an internal frame). The equations for relative motion, discussed in **Motion in Two and Three Dimensions**, provide a link to calculating the kinetic energy of an object with respect to different frames of reference.

Example 7.7

Kinetic Energy Relative to Different Frames

A 75.0-kg person walks down the central aisle of a subway car at a speed of 1.50 m/s relative to the car, whereas the train is moving at 15.0 m/s relative to the tracks. (a) What is the person’s kinetic energy relative to the car? (b) What is the person’s kinetic energy relative to the tracks? (c) What is the person’s kinetic energy relative to a frame moving with the person?

Strategy

Since speeds are given, we can use $\frac{1}{2}mv^2$ to calculate the person’s kinetic energy. However, in part (a), the person’s speed is relative to the subway car (as given); in part (b), it is relative to the tracks; and in part (c), it is zero. If we denote the car frame by C, the track frame by T, and the person by P, the relative velocities in part (b) are related by $\vec{v}_{\text{PT}} = \vec{v}_{\text{PC}} + \vec{v}_{\text{CT}}$. We can assume that the central aisle and the tracks lie along the same line, but the direction the person is walking relative to the car isn’t specified, so we will give an answer for each possibility, $v_{\text{PT}} = v_{\text{CT}} \pm v_{\text{PC}}$, as shown in **Figure 7.10**.

Figure 7.10 The possible motions of a person walking in a train are (a) toward the front of the car and (b) toward the back of the car.

Solution

a. $K = \frac{1}{2}(75.0\,\text{kg})(1.50\,\text{m/s})^2 = 84.4\,\text{J}.$

b. $v_{PT} = (15.0 \pm 1.50)\,\text{m/s}.$ Therefore, the two possible values for kinetic energy relative to the car are

$$K = \frac{1}{2}(75.0\,\text{kg})(13.5\,\text{m/s})^2 = 6.83\,\text{kJ}$$

and

$$K = \frac{1}{2}(75.0\,\text{kg})(16.5\,\text{m/s})^2 = 10.2\,\text{kJ}.$$

c. In a frame where $v_P = 0$, $K = 0$ as well.

Significance

You can see that the kinetic energy of an object can have very different values, depending on the frame of reference. However, the kinetic energy of an object can never be negative, since it is the product of the mass and the square of the speed, both of which are always positive or zero.

7.6 Check Your Understanding You are rowing a boat parallel to the banks of a river. Your kinetic energy relative to the banks is less than your kinetic energy relative to the water. Are you rowing with or against the current?

The kinetic energy of a particle is a single quantity, but the kinetic energy of a system of particles can sometimes be divided into various types, depending on the system and its motion. For example, if all the particles in a system have the same velocity, the system is undergoing translational motion and has translational kinetic energy. If an object is rotating, it could have rotational kinetic energy, or if it's vibrating, it could have vibrational kinetic energy. The kinetic energy of a system, relative to an internal frame of reference, may be called internal kinetic energy. The kinetic energy associated with random molecular motion may be called thermal energy. These names will be used in later chapters of the book, when appropriate. Regardless of the name, every kind of kinetic energy is the same physical quantity, representing energy associated with motion.

Example 7.8

Special Names for Kinetic Energy

(a) A player lobs a mid-court pass with a 624-g basketball, which covers 15 m in 2 s. What is the basketball's horizontal translational kinetic energy while in flight? (b) An average molecule of air, in the basketball in part (a), has a mass of 29 u, and an average speed of 500 m/s, relative to the basketball. There are about 3×10^{23} molecules inside it, moving in random directions, when the ball is properly inflated. What is the average translational kinetic energy of the random motion of all the molecules inside, relative to the basketball? (c) How fast would the basketball have to travel relative to the court, as in part (a), so as to have a kinetic energy equal to the amount in part (b)?

Strategy

In part (a), first find the horizontal speed of the basketball and then use the definition of kinetic energy in terms of mass and speed, $K = \frac{1}{2}mv^2$. Then in part (b), convert unified units to kilograms and then use $K = \frac{1}{2}mv^2$ to get the average translational kinetic energy of one molecule, relative to the basketball. Then multiply by the number of molecules to get the total result. Finally, in part (c), we can substitute the amount of kinetic energy in part (b), and the mass of the basketball in part (a), into the definition $K = \frac{1}{2}mv^2$, and solve for *v*.

Solution

a. The horizontal speed is (15 m)/(2 s), so the horizontal kinetic energy of the basketball is

$$\frac{1}{2}(0.624\ \text{kg})(7.5\ \text{m/s})^2 = 17.6\ \text{J}.$$

b. The average translational kinetic energy of a molecule is

$$\frac{1}{2}(29\ \text{u})(1.66 \times 10^{-27}\ \text{kg/u})(500\ \text{m/s})^2 = 6.02 \times 10^{-21}\ \text{J},$$

and the total kinetic energy of all the molecules is

$$(3 \times 10^{23})(6.02 \times 10^{-21}\ \text{J}) = 1.80\ \text{kJ}.$$

c. $v = \sqrt{2(1.8\ \text{kJ})/(0.624\ \text{kg})} = 76.0\ \text{m/s}.$

Significance

In part (a), this kind of kinetic energy can be called the horizontal kinetic energy of an object (the basketball), relative to its surroundings (the court). If the basketball were spinning, all parts of it would have not just the average speed, but it would also have rotational kinetic energy. Part (b) reminds us that this kind of kinetic energy can be called internal or thermal kinetic energy. Notice that this energy is about a hundred times the energy in part (a). How to make use of thermal energy will be the subject of the chapters on thermodynamics. In part (c), since the energy in part (b) is about 100 times that in part (a), the speed should be about 10 times as big, which it is (76 compared to 7.5 m/s).

7.3 | Work-Energy Theorem

Learning Objectives

By the end of this section, you will be able to:

- Apply the work-energy theorem to find information about the motion of a particle, given the forces acting on it
- Use the work-energy theorem to find information about the forces acting on a particle, given information about its motion

We have discussed how to find the work done on a particle by the forces that act on it, but how is that work manifested in the motion of the particle? According to Newton's second law of motion, the sum of all the forces acting on a particle, or the net force, determines the rate of change in the momentum of the particle, or its motion. Therefore, we should consider the work done by all the forces acting on a particle, or the **net work**, to see what effect it has on the particle's motion.

Let's start by looking at the net work done on a particle as it moves over an infinitesimal displacement, which is the dot product of the net force and the displacement: $dW_{\text{net}} = \vec{\mathbf{F}}_{\text{net}} \cdot d\vec{\mathbf{r}}$. Newton's second law tells us that $\vec{\mathbf{F}}_{\text{net}} = m(d\vec{\mathbf{v}}/dt),$, so $dW_{\text{net}} = m(d\vec{\mathbf{v}}/dt) \cdot d\vec{\mathbf{r}}$. For the mathematical functions describing the motion of a physical particle, we can rearrange the differentials *dt*, etc., as algebraic quantities in this expression, that is,

$$dW_{\text{net}} = m\left(\frac{d\vec{\mathbf{v}}}{dt}\right) \cdot d\vec{\mathbf{r}} = md\vec{\mathbf{v}} \cdot \left(\frac{d\vec{\mathbf{r}}}{dt}\right) = m\vec{\mathbf{v}} \cdot d\vec{\mathbf{v}},$$

where we substituted the velocity for the time derivative of the displacement and used the commutative property of the dot product [**Equation 2.30**]. Since derivatives and integrals of scalars are probably more familiar to you at this point, we express the dot product in terms of Cartesian coordinates before we integrate between any two points A and B on the particle's trajectory. This gives us the net work done on the particle:

$$\begin{aligned} W_{\text{net, }AB} &= \int_A^B (mv_x dv_x + mv_y dv_y + mv_z dv_z) \\ &= \frac{1}{2}m\left|v_x^2 + v_y^2 + v_z^2\right|_A^B = \left|\frac{1}{2}mv^2\right|_A^B = K_B - K_A. \end{aligned} \tag{7.8}$$

In the middle step, we used the fact that the square of the velocity is the sum of the squares of its Cartesian components, and in the last step, we used the definition of the particle's kinetic energy. This important result is called the **work-energy theorem** (**Figure 7.11**).

Work-Energy Theorem

The net work done on a particle equals the change in the particle's kinetic energy:

$$W_{\text{net}} = K_B - K_A. \tag{7.9}$$

Figure 7.11 Horse pulls are common events at state fairs. The work done by the horses pulling on the load results in a change in kinetic energy of the load, ultimately going faster. (credit: "Jassen"/ Flickr)

According to this theorem, when an object slows down, its final kinetic energy is less than its initial kinetic energy, the change in its kinetic energy is negative, and so is the net work done on it. If an object speeds up, the net work done on it is positive. When calculating the net work, you must include all the forces that act on an object. If you leave out any forces that act on an object, or if you include any forces that don't act on it, you will get a wrong result.

The importance of the work-energy theorem, and the further generalizations to which it leads, is that it makes some types of calculations much simpler to accomplish than they would be by trying to solve Newton's second law. For example, in **Newton's Laws of Motion**, we found the speed of an object sliding down a frictionless plane by solving Newton's second law for the acceleration and using kinematic equations for constant acceleration, obtaining

$$v_{\text{f}}^2 = v_{\text{i}}^2 + 2g(s_{\text{f}} - s_{\text{i}})\sin\theta,$$

where s is the displacement down the plane.

We can also get this result from the work-energy theorem. Since only two forces are acting on the object—gravity and the normal force—and the normal force doesn't do any work, the net work is just the work done by gravity. This only depends on the object's weight and the difference in height, so

$$W_{\text{net}} = W_{\text{grav}} = -mg(y_{\text{f}} - y_{\text{i}}),$$

where y is positive up. The work-energy theorem says that this equals the change in kinetic energy:

$$-mg(y_{\text{f}} - y_{\text{i}}) = \frac{1}{2}m(v_{\text{f}}^2 - v_{\text{i}}^2).$$

Using a right triangle, we can see that $(y_{\text{f}} - y_{\text{i}}) = (s_{\text{f}} - s_{\text{i}})\sin\theta,$ so the result for the final speed is the same.

What is gained by using the work-energy theorem? The answer is that for a frictionless plane surface, not much. However, Newton's second law is easy to solve only for this particular case, whereas the work-energy theorem gives the final speed for any shaped frictionless surface. For an arbitrary curved surface, the normal force is not constant, and Newton's second law may be difficult or impossible to solve analytically. Constant or not, for motion along a surface, the normal force never does any work, because it's perpendicular to the displacement. A calculation using the work-energy theorem avoids this difficulty and applies to more general situations.

Problem-Solving Strategy: Work-Energy Theorem

1. Draw a free-body diagram for each force on the object.
2. Determine whether or not each force does work over the displacement in the diagram. Be sure to keep any positive or negative signs in the work done.
3. Add up the total amount of work done by each force.
4. Set this total work equal to the change in kinetic energy and solve for any unknown parameter.
5. Check your answers. If the object is traveling at a constant speed or zero acceleration, the total work done should be zero and match the change in kinetic energy. If the total work is positive, the object must have sped up or increased kinetic energy. If the total work is negative, the object must have slowed down or decreased kinetic energy.

Example 7.9

Loop-the-Loop

The frictionless track for a toy car includes a loop-the-loop of radius R. How high, measured from the bottom of the loop, must the car be placed to start from rest on the approaching section of track and go all the way around the loop?

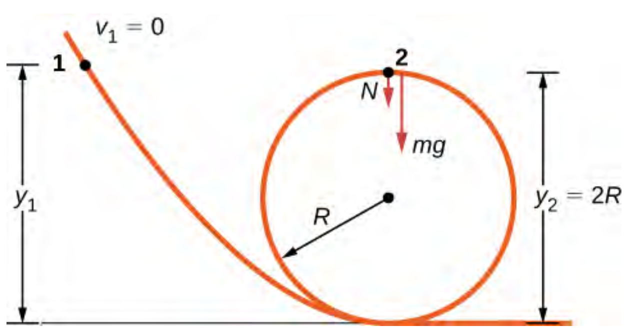


Figure 7.12 A frictionless track for a toy car has a loop-the-loop in it. How high must the car start so that it can go around the loop without falling off?

Strategy

The free-body diagram at the final position of the object is drawn in **Figure 7.12**. The gravitational work is the only work done over the displacement that is not zero. Since the weight points in the same direction as the net vertical displacement, the total work done by the gravitational force is positive. From the work-energy theorem, the starting height determines the speed of the car at the top of the loop,

$$mg(y_2 - y_1) = \frac{1}{2}mv_2{}^2,$$

where the notation is shown in the accompanying figure. At the top of the loop, the normal force and gravity are both down and the acceleration is centripetal, so

$$a_{\text{top}} = \frac{F}{m} = \frac{N + mg}{m} = \frac{v_2^2}{R}.$$

The condition for maintaining contact with the track is that there must be some normal force, however slight; that is, $N > 0$. Substituting for v_2^2 and *N*, we can find the condition for y_1.

Solution

Implement the steps in the strategy to arrive at the desired result:

$$N = \frac{-mg + mv_2^2}{R} = \frac{-mg + 2mg(y_1 - 2R)}{R} > 0 \quad \text{or} \quad y_1 > \frac{5R}{2}.$$

Significance

On the surface of the loop, the normal component of gravity and the normal contact force must provide the centripetal acceleration of the car going around the loop. The tangential component of gravity slows down or speeds up the car. A child would find out how high to start the car by trial and error, but now that you know the work-energy theorem, you can predict the minimum height (as well as other more useful results) from physical principles. By using the work-energy theorem, you did not have to solve a differential equation to determine the height.

7.7 Check Your Understanding Suppose the radius of the loop-the-loop in **Example 7.9** is 15 cm and the toy car starts from rest at a height of 45 cm above the bottom. What is its speed at the top of the loop?

Visit Carleton College's site to see a **video (https://openstaxcollege.org/l/21carcollvidrol)** of a looping rollercoaster.

In situations where the motion of an object is known, but the values of one or more of the forces acting on it are not known, you may be able to use the work-energy theorem to get some information about the forces. Work depends on the force and the distance over which it acts, so the information is provided via their product.

Example 7.10

Determining a Stopping Force

A bullet from a 0.22LR-caliber cartridge has a mass of 40 grains (2.60 g) and a muzzle velocity of 1100 ft./s (335 m/s). It can penetrate eight 1-inch pine boards, each with thickness 0.75 inches. What is the average stopping force exerted by the wood, as shown in **Figure 7.13**?

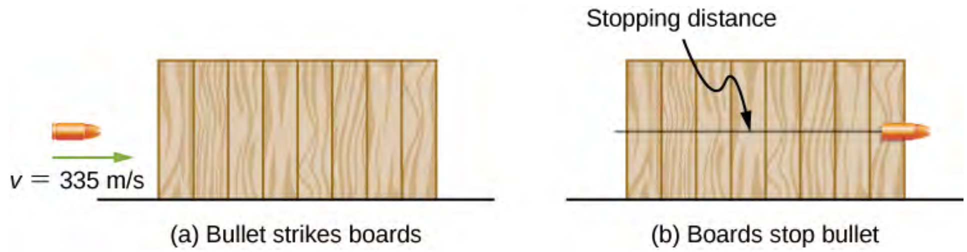


Figure 7.13 The boards exert a force to stop the bullet. As a result, the boards do work and the bullet loses kinetic energy.

Strategy

We can assume that under the general conditions stated, the bullet loses all its kinetic energy penetrating the boards, so the work-energy theorem says its initial kinetic energy is equal to the average stopping force times the distance penetrated. The change in the bullet's kinetic energy and the net work done stopping it are both negative, so when you write out the work-energy theorem, with the net work equal to the average force times the stopping distance, that's what you get. The total thickness of eight 1-inch pine boards that the bullet penetrates is $8 \times \frac{3}{4}\,\text{in.} = 6\,\text{in.} = 15.2\,\text{cm}.$

Solution

Applying the work-energy theorem, we get

$$W_{\text{net}} = -F_{\text{ave}}\,\Delta s_{\text{stop}} = -K_{\text{initial}},$$

so

$$F_{\text{ave}} = \frac{\frac{1}{2}mv^2}{\Delta s_{\text{stop}}} = \frac{\frac{1}{2}(2.6 \times 10^{-3}\,\text{kg})(335\,\text{m/s})^2}{0.152\,\text{m}} = 960\,\text{N}.$$

Significance

We could have used Newton's second law and kinematics in this example, but the work-energy theorem also supplies an answer to less simple situations. The penetration of a bullet, fired vertically upward into a block of wood, is discussed in one section of Asif Shakur's recent article ["Bullet-Block Science Video Puzzle." *The Physics Teacher* (January 2015) 53(1): 15-16]. If the bullet is fired dead center into the block, it loses all its kinetic energy and penetrates slightly farther than if fired off-center. The reason is that if the bullet hits off-center, it has a little kinetic energy after it stops penetrating, because the block rotates. The work-energy theorem implies that a smaller change in kinetic energy results in a smaller penetration. You will understand more of the physics in this interesting article after you finish reading **Angular Momentum**.

Learn more about work and energy in this **PhET simulation (https://openstaxcollege.org/l/21PhETSimRamp)** called "the ramp." Try changing the force pushing the box and the frictional force along the incline. The work and energy plots can be examined to note the total work done and change in kinetic energy of the box.

7.4 | Power

Learning Objectives

By the end of this section, you will be able to:

- Relate the work done during a time interval to the power delivered
- Find the power expended by a force acting on a moving body

The concept of work involves force and displacement; the work-energy theorem relates the net work done on a body to the difference in its kinetic energy, calculated between two points on its trajectory. None of these quantities or relations involves time explicitly, yet we know that the time available to accomplish a particular amount of work is frequently just as important to us as the amount itself. In the chapter-opening figure, several sprinters may have achieved the same velocity at the finish, and therefore did the same amount of work, but the winner of the race did it in the least amount of time.

We express the relation between work done and the time interval involved in doing it, by introducing the concept of power. Since work can vary as a function of time, we first define **average power** as the work done during a time interval, divided by the interval,

$$P_{\text{ave}} = \frac{\Delta W}{\Delta t}. \quad (7.10)$$

Then, we can define the **instantaneous power** (frequently referred to as just plain **power**).

Power

Power is defined as the rate of doing work, or the limit of the average power for time intervals approaching zero,

$$P = \frac{dW}{dt}. \quad (7.11)$$

If the power is constant over a time interval, the average power for that interval equals the instantaneous power, and the work done by the agent supplying the power is $W = P\Delta t$. If the power during an interval varies with time, then the work done is the time integral of the power,

$$W = \int P dt.$$

The work-energy theorem relates how work can be transformed into kinetic energy. Since there are other forms of energy as well, as we discuss in the next chapter, we can also define power as the rate of transfer of energy. Work and energy are measured in units of joules, so power is measured in units of joules per second, which has been given the SI name watts, abbreviation W: $1 \text{ J/s} = 1 \text{ W}$. Another common unit for expressing the power capability of everyday devices is horsepower: $1 \text{ hp} = 746 \text{ W}$.

Example 7.11

Pull-Up Power

An 80-kg army trainee does 10 pull-ups in 10 s (**Figure 7.14**). How much average power do the trainee's muscles supply moving his body? (*Hint:* Make reasonable estimates for any quantities needed.)

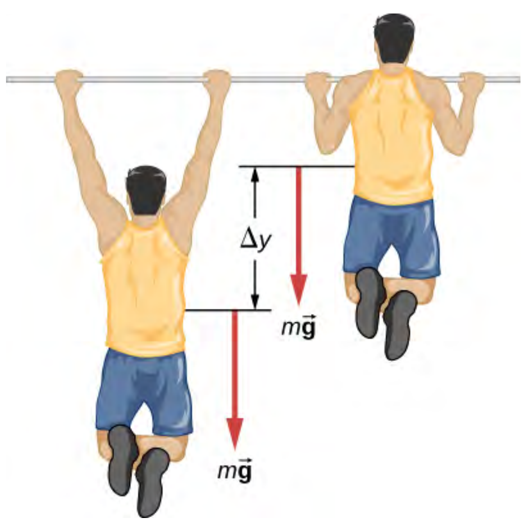


Figure 7.14 What is the power expended in doing ten pull-ups in ten seconds?

Strategy

The work done against gravity, going up or down a distance Δy, is $mg\Delta y$. (If you lift and lower yourself at constant speed, the force you exert cancels gravity over the whole pull-up cycle.) Thus, the work done by the trainee's muscles (moving, but not accelerating, his body) for a complete repetition (up and down) is $2mg\Delta y$. Let's assume that $\Delta y = 2\text{ft} \approx 60\,\text{cm}$. Also, assume that the arms comprise 10% of the body mass and are not included in the moving mass. With these assumptions, we can calculate the work done for 10 pull-ups and divide by 10 s to get the average power.

Solution

The result we get, applying our assumptions, is

$$P_{\text{ave}} = \frac{10 \times 2(0.9 \times 80\,\text{kg})(9.8\,\text{m/s}^2)(0.6\,\text{m})}{10\,\text{s}} = 850\,\text{W}.$$

Significance

This is typical for power expenditure in strenuous exercise; in everyday units, it's somewhat more than one horsepower $(1\,\text{hp} = 746\,\text{W})$.

7.8 Check Your Understanding Estimate the power expended by a weightlifter raising a 150-kg barbell 2 m in 3 s.

The power involved in moving a body can also be expressed in terms of the forces acting on it. If a force $\vec{\mathbf{F}}$ acts on a body that is displaced $d\vec{\mathbf{r}}$ in a time *dt*, the power expended by the force is

$$P = \frac{dW}{dt} = \frac{\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \left(\frac{d\vec{\mathbf{r}}}{dt}\right) = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}, \tag{7.12}$$

where $\vec{\mathbf{v}}$ is the velocity of the body. The fact that the limits implied by the derivatives exist, for the motion of a real body, justifies the rearrangement of the infinitesimals.

Example 7.12

Automotive Power Driving Uphill

How much power must an automobile engine expend to move a 1200-kg car up a 15% grade at 90 km/h (Figure 7.15)? Assume that 25% of this power is dissipated overcoming air resistance and friction.

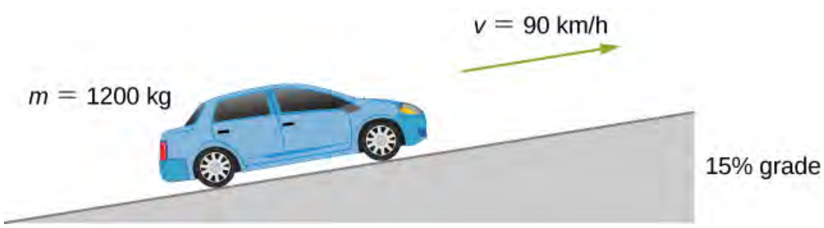


Figure 7.15 We want to calculate the power needed to move a car up a hill at constant speed.

Strategy

At constant velocity, there is no change in kinetic energy, so the net work done to move the car is zero. Therefore the power supplied by the engine to move the car equals the power expended against gravity and air resistance. By assumption, 75% of the power is supplied against gravity, which equals $m\,\vec{\mathbf{g}} \cdot \vec{\mathbf{v}} = mgv\sin\theta,$ where θ is the angle of the incline. A 15% grade means $\tan\theta = 0.15.$ This reasoning allows us to solve for the power required.

Solution

Carrying out the suggested steps, we find

$$0.75\,P = mgv\,\sin(\tan^{-1}\,0.15),$$

or

$$P = \frac{(1200 \times 9.8\,\text{N})(90\,\text{m}/3.6\,\text{s})\sin(8.53°)}{0.75} = 58\,\text{kW},$$

or about 78 hp. (You should supply the steps used to convert units.)

Significance

This is a reasonable amount of power for the engine of a small to mid-size car to supply $(1\,\text{hp} = 0.746\,\text{kW})$.

Note that this is only the power expended to move the car. Much of the engine's power goes elsewhere, for example, into waste heat. That's why cars need radiators. Any remaining power could be used for acceleration, or to operate the car's accessories.

CHAPTER 7 REVIEW

KEY TERMS

average power work done in a time interval divided by the time interval

kinetic energy energy of motion, one-half an object's mass times the square of its speed

net work work done by all the forces acting on an object

power (or instantaneous power) rate of doing work

work done when a force acts on something that undergoes a displacement from one position to another

work done by a force integral, from the initial position to the final position, of the dot product of the force and the infinitesimal displacement along the path over which the force acts

work-energy theorem net work done on a particle is equal to the change in its kinetic energy

KEY EQUATIONS

Work done by a force over an infinitesimal displacement	$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \lvert\vec{\mathbf{F}}\rvert\lvert d\vec{\mathbf{r}}\rvert \cos\theta$
Work done by a force acting along a path from *A* to *B*	$W_{AB} = \int_{\text{path}AB} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$
Work done by a constant force of kinetic friction	$W_{\text{fr}} = -f_k \lvert l_{AB}\rvert$
Work done going from *A* to *B* by Earth's gravity, near its surface	$W_{\text{grav},AB} = -mg(y_B - y_A)$
Work done going from *A* to *B* by one-dimensional spring force	$W_{\text{spring},AB} = -\left(\frac{1}{2}k\right)\left(x_B^2 - x_A^2\right)$
Kinetic energy of a non-relativistic particle	$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$
Work-energy theorem	$W_{\text{net}} = K_B - K_A$
Power as rate of doing work	$P = \frac{dW}{dt}$
Power as the dot product of force and velocity	$P = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$

SUMMARY

7.1 Work

- The infinitesimal increment of work done by a force, acting over an infinitesimal displacement, is the dot product of the force and the displacement.
- The work done by a force, acting over a finite path, is the integral of the infinitesimal increments of work done along the path.
- The work done *against* a force is the negative of the work done *by* the force.
- The work done by a normal or frictional contact force must be determined in each particular case.
- The work done by the force of gravity, on an object near the surface of Earth, depends only on the weight of the object and the difference in height through which it moved.
- The work done by a spring force, acting from an initial position to a final position, depends only on the spring constant and the squares of those positions.

7.2 Kinetic Energy

- The kinetic energy of a particle is the product of one-half its mass and the square of its speed, for non-relativistic speeds.
- The kinetic energy of a system is the sum of the kinetic energies of all the particles in the system.
- Kinetic energy is relative to a frame of reference, is always positive, and is sometimes given special names for different types of motion.

7.3 Work-Energy Theorem

- Because the net force on a particle is equal to its mass times the derivative of its velocity, the integral for the net work done on the particle is equal to the change in the particle's kinetic energy. This is the work-energy theorem.
- You can use the work-energy theorem to find certain properties of a system, without having to solve the differential equation for Newton's second law.

7.4 Power

- Power is the rate of doing work; that is, the derivative of work with respect to time.
- Alternatively, the work done, during a time interval, is the integral of the power supplied over the time interval.
- The power delivered by a force, acting on a moving particle, is the dot product of the force and the particle's velocity.

CONCEPTUAL QUESTIONS

7.1 Work

1. Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

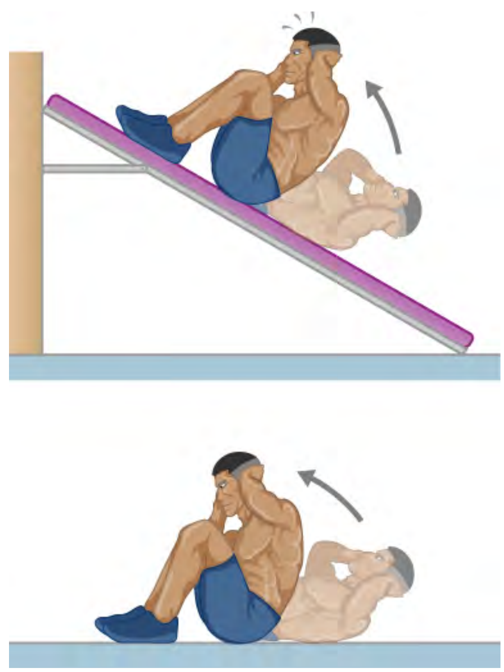
2. Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

3. Describe a situation in which a force is exerted for a long time but does no work. Explain.

4. A body moves in a circle at constant speed. Does the centripetal force that accelerates the body do any work? Explain.

5. Suppose you throw a ball upward and catch it when it returns at the same height. How much work does the gravitational force do on the ball over its entire trip?

6. Why is it more difficult to do sit-ups while on a slant board than on a horizontal surface? (See below.)



7. As a young man, Tarzan climbed up a vine to reach his tree house. As he got older, he decided to build and use a staircase instead. Since the work of the gravitational force *mg* is path independent, what did the King of the Apes gain in using stairs?

7.2 Kinetic Energy

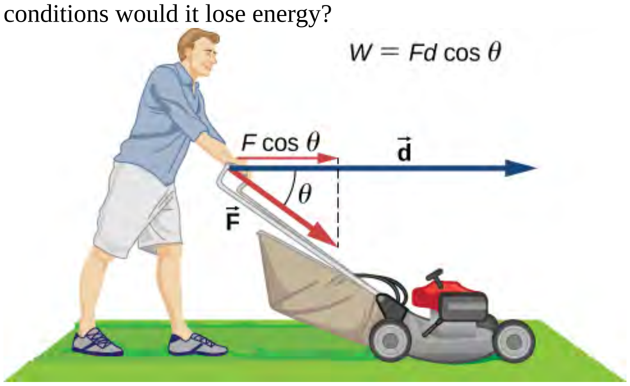
8. A particle of m has a velocity of $v_x\,\hat{\mathbf{i}} + v_y\,\hat{\mathbf{j}} + v_z\,\hat{\mathbf{k}}$. Is its kinetic energy given by $m(v_x{}^2\,\hat{\mathbf{i}} + v_y{}^2\,\hat{\mathbf{j}} + v_z{}^2\,\hat{\mathbf{k}})/2$? If not, what is the correct expression?

9. One particle has mass m and a second particle has mass $2m$. The second particle is moving with speed v and the first with speed $2v$. How do their kinetic energies compare?

10. A person drops a pebble of mass m_1 from a height h, and it hits the floor with kinetic energy K. The person drops another pebble of mass m_2 from a height of $2h$, and it hits the floor with the same kinetic energy K. How do the masses of the pebbles compare?

7.3 Work-Energy Theorem

11. The person shown below does work on the lawn mower. Under what conditions would the mower gain energy from the person pushing the mower? Under what conditions would it lose energy?



12. Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

13. Two marbles of masses m and $2m$ are dropped from a height h. Compare their kinetic energies when they reach the ground.

14. Compare the work required to accelerate a car of mass 2000 kg from 30.0 to 40.0 km/h with that required for an acceleration from 50.0 to 60.0 km/h.

15. Suppose you are jogging at constant velocity. Are you doing any work on the environment and vice versa?

16. Two forces act to double the speed of a particle, initially moving with kinetic energy of 1 J. One of the forces does 4 J of work. How much work does the other force do?

7.4 Power

17. Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

18. Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

19. A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

20. Does the work done in lifting an object depend on how fast it is lifted? Does the power expended depend on how fast it is lifted?

21. Can the power expended by a force be negative?

22. How can a 50-W light bulb use more energy than a 1000-W oven?

PROBLEMS

7.1 Work

23. How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N?

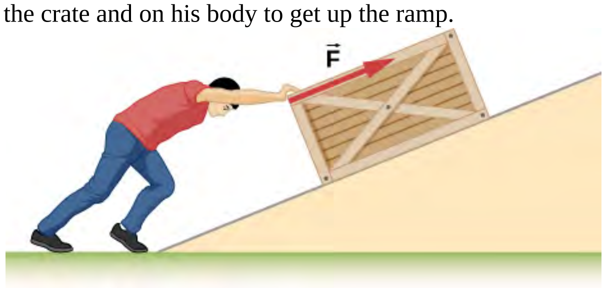
24. A 75.0-kg person climbs stairs, gaining 2.50 m in height. Find the work done to accomplish this task.

25. (a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

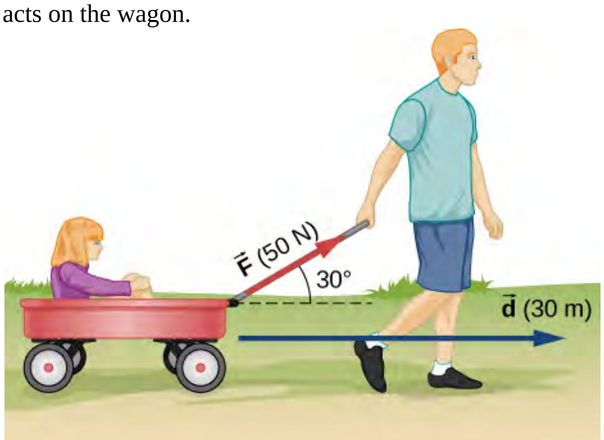
26. Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (The energy content of gasoline is about 140 MJ/gal.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed?

(b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

27. Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of $20.0°$ with the horizontal (see below). He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate and on his body to get up the ramp.

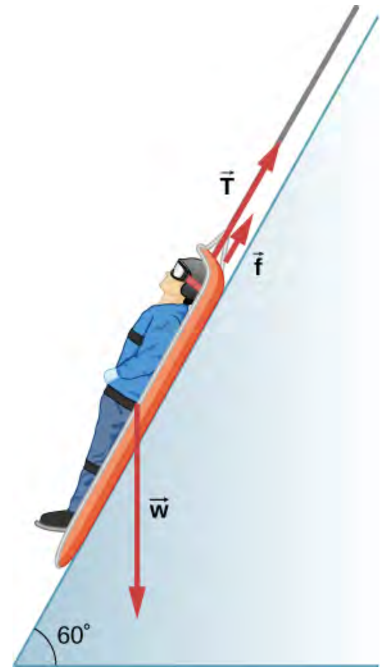


28. How much work is done by the boy pulling his sister 30.0 m in a wagon as shown below? Assume no friction acts on the wagon.



29. A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction $25.0°$ below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

30. Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a $60.0°$ slope at constant speed, as shown below. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?



31. A constant 20-N force pushes a small ball in the direction of the force over a distance of 5.0 m. What is the work done by the force?

32. A toy cart is pulled a distance of 6.0 m in a straight line across the floor. The force pulling the cart has a magnitude of 20 N and is directed at $37°$ above the horizontal. What is the work done by this force?

33. A 5.0-kg box rests on a horizontal surface. The coefficient of kinetic friction between the box and surface is $\mu_K = 0.50.$ A horizontal force pulls the box at constant velocity for 10 cm. Find the work done by (a) the applied horizontal force, (b) the frictional force, and (c) the net force.

34. A sled plus passenger with total mass 50 kg is pulled 20 m across the snow $(\mu_k = 0.20)$ at constant velocity by a force directed $25°$ above the horizontal. Calculate (a) the work of the applied force, (b) the work of friction, and (c) the total work.

35. Suppose that the sled plus passenger of the preceding problem is pushed 20 m across the snow at constant velocity by a force directed $30°$ below the horizontal. Calculate (a) the work of the applied force, (b) the work of friction, and (c) the total work.

36. How much work does the force $F(x) = (-2.0/x)\,\text{N}$ do on a particle as it moves from $x = 2.0\,\text{m}$ to $x = 5.0\,\text{m}$?

37. How much work is done against the gravitational force on a 5.0-kg briefcase when it is carried from the ground floor to the roof of the Empire State Building, a vertical climb of 380 m?

38. It takes 500 J of work to compress a spring 10 cm. What is the force constant of the spring?

39. A bungee cord is essentially a very long rubber band that can stretch up to four times its unstretched length. However, its spring constant varies over its stretch [see Menz, P.G. "The Physics of Bungee Jumping." *The Physics Teacher* (November 1993) 31: 483-487]. Take the length of the cord to be along the *x*-direction and define the stretch *x* as the length of the cord *l* minus its un-stretched length l_0; that is, $x = l - l_0$ (see below). Suppose a particular bungee cord has a spring constant, for $0 \le x \le 4.88\text{ m}$, of $k_1 = 204\text{ N/m}$ and for $4.88\text{ m} \le x$, of $k_2 = 111\text{ N/m}$. (Recall that the spring constant is the slope of the force *F(x)* versus its stretch *x*.) (a) What is the tension in the cord when the stretch is 16.7 m (the maximum desired for a given jump)? (b) How much work must be done against the elastic force of the bungee cord to stretch it 16.7 m?

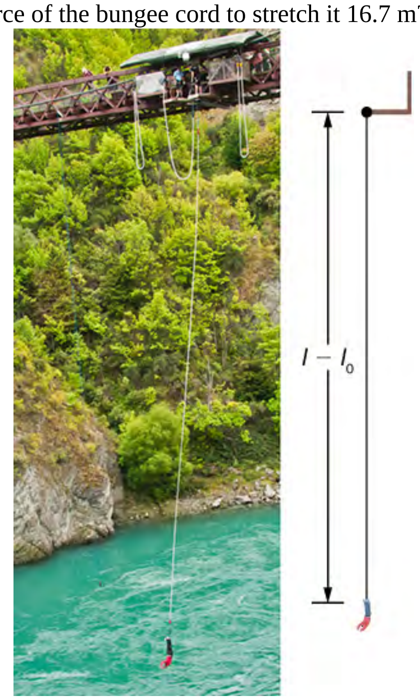


Figure 7.16 (credit: Graeme Churchard)

40. A bungee cord exerts a nonlinear elastic force of magnitude $F(x) = k_1 x + k_2 x^3$, where *x* is the distance the cord is stretched, $k_1 = 204\text{ N/m}$ and $k_2 = -0.233\text{ N/m}^3$. How much work must be done on the cord to stretch it 16.7 m?

41. Engineers desire to model the magnitude of the elastic force of a bungee cord using the equation

$$F(x) = a\left[\frac{x + 9\text{ m}}{9\text{ m}} - \left(\frac{9\text{ m}}{x + 9\text{ m}}\right)^2\right],$$

where *x* is the stretch of the cord along its length and *a* is a constant. If it takes 22.0 kJ of work to stretch the cord by 16.7 m, determine the value of the constant *a*.

42. A particle moving in the *xy*-plane is subject to a force

$$\vec{\mathbf{F}}(x, y) = (50\text{ N}\cdot\text{m}^2)\frac{(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}})}{(x^2 + y^2)^{3/2}},$$

where *x* and *y* are in meters. Calculate the work done on the particle by this force, as it moves in a straight line from the point (3 m, 4 m) to the point (8 m, 6 m).

43. A particle moves along a curved path $y(x) = (10\text{ m})\{1 + \cos[(0.1\text{ m}^{-1})x]\}$, from $x = 0$ to $x = 10\pi\text{ m}$, subject to a tangential force of variable magnitude $F(x) = (10\text{ N})\sin[(0.1\text{ m}^{-1})x]$. How much work does the force do? (*Hint:* Consult a table of integrals or use a numerical integration program.)

7.2 Kinetic Energy

44. Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

45. (a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

46. Estimate the kinetic energy of a 90,000-ton aircraft carrier moving at a speed of at 30 knots. You will need to look up the definition of a nautical mile to use in converting the unit for speed, where 1 knot equals 1 nautical mile per hour.

47. Calculate the kinetic energies of (a) a 2000.0-kg automobile moving at 100.0 km/h; (b) an 80.-kg runner sprinting at 10. m/s; and (c) a 9.1×10^{-31} -kg electron moving at 2.0×10^7 m/s.

48. A 5.0-kg body has three times the kinetic energy of an 8.0-kg body. Calculate the ratio of the speeds of these bodies.

49. An 8.0-g bullet has a speed of 800 m/s. (a) What is its kinetic energy? (b) What is its kinetic energy if the speed is halved?

7.3 Work-Energy Theorem

50. (a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

51. A car's bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.

52. Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent's face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the gory old days when no gloves were used, and the knuckles and face would compress only 2.00 cm. Assume the change in mass by removing the glove is negligible. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

53. Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

54. A 5.0-kg box has an acceleration of $2.0\ \text{m/s}^2$ when it is pulled by a horizontal force across a surface with $\mu_K = 0.50.$ Find the work done over a distance of 10 cm by (a) the horizontal force, (b) the frictional force, and (c) the net force. (d) What is the change in kinetic energy of the box?

55. A constant 10-N horizontal force is applied to a 20-kg cart at rest on a level floor. If friction is negligible, what is the speed of the cart when it has been pushed 8.0 m?

56. In the preceding problem, the 10-N force is applied at an angle of $45°$ below the horizontal. What is the speed of the cart when it has been pushed 8.0 m?

57. Compare the work required to stop a 100-kg crate sliding at 1.0 m/s and an 8.0-g bullet traveling at 500 m/s.

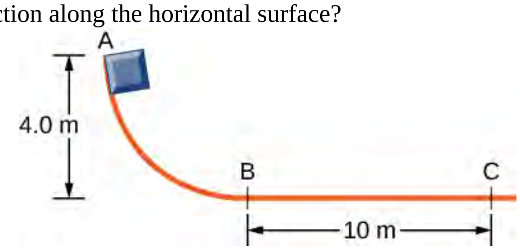
58. A wagon with its passenger sits at the top of a hill. The wagon is given a slight push and rolls 100 m down a $10°$ incline to the bottom of the hill. What is the wagon's speed when it reaches the end of the incline. Assume that the retarding force of friction is negligible.

59. An 8.0-g bullet with a speed of 800 m/s is shot into a wooden block and penetrates 20 cm before stopping. What is the average force of the wood on the bullet? Assume the block does not move.

60. A 2.0-kg block starts with a speed of 10 m/s at the bottom of a plane inclined at $37°$ to the horizontal. The coefficient of sliding friction between the block and plane is $\mu_k = 0.30.$ (a) Use the work-energy principle to determine how far the block slides along the plane before momentarily coming to rest. (b) After stopping, the block slides back down the plane. What is its speed when it reaches the bottom? (*Hint:* For the round trip, only the force of friction does work on the block.)

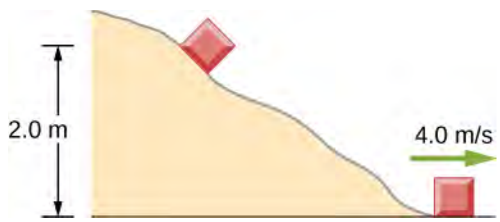
61. When a 3.0-kg block is pushed against a massless spring of force constant constant $4.5 \times 10^3\ \text{N/m},$ the spring is compressed 8.0 cm. The block is released, and it slides 2.0 m (from the point at which it is released) across a horizontal surface before friction stops it. What is the coefficient of kinetic friction between the block and the surface?

62. A small block of mass 200 g starts at rest at A, slides to B where its speed is $v_B = 8.0\ \text{m/s},$ then slides along the horizontal surface a distance 10 m before coming to rest at C. (See below.) (a) What is the work of friction along the curved surface? (b) What is the coefficient of kinetic friction along the horizontal surface?



63. A small object is placed at the top of an incline that is essentially frictionless. The object slides down the incline onto a rough horizontal surface, where it stops in 5.0 s after traveling 60 m. (a) What is the speed of the object at the bottom of the incline and its acceleration along the horizontal surface? (b) What is the height of the incline?

64. When released, a 100-g block slides down the path shown below, reaching the bottom with a speed of 4.0 m/s. How much work does the force of friction do?



65. A 0.22LR-caliber bullet like that mentioned in **Example 7.10** is fired into a door made of a single thickness of 1-inch pine boards. How fast would the bullet be traveling after it penetrated through the door?

66. A sled starts from rest at the top of a snow-covered incline that makes a $22°$ angle with the horizontal. After sliding 75 m down the slope, its speed is 14 m/s. Use the work-energy theorem to calculate the coefficient of kinetic friction between the runners of the sled and the snowy surface.

7.4 Power

67. A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

68. What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is \$0.0900 per $\text{kW} \cdot \text{h}$?

69. A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is \$0.110 per $\text{kW} \cdot \text{h}$?

70. (a) What is the average power consumption in watts of an appliance that uses 5.00 $\text{kW} \cdot \text{h}$ of energy per day? (b) How many joules of energy does this appliance consume in a year?

71. (a) What is the average useful power output of a person who does 6.00×10^6 J of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

72. A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?

73. (a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp equals 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m high hill in the process?

74. (a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is \$0.0900 per $\text{kW} \cdot \text{h}$?

75. (a) How long would it take a 1.50×10^5-kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (*Hint:* You must find the distance the plane travels in 1200 s assuming constant acceleration.)

76. Calculate the power output needed for a 950-kg car to climb a $2.00°$ slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N.

77. A man of mass 80 kg runs up a flight of stairs 20 m high in 10 s. (a) how much power is used to lift the man? (b) If the man's body is 25% efficient, how much power does he expend?

78. The man of the preceding problem consumes approximately 1.05×10^7 J (2500 food calories) of energy per day in maintaining a constant weight. What is the average power he produces over a day? Compare this with his power production when he runs up the stairs.

79. An electron in a television tube is accelerated uniformly from rest to a speed of 8.4×10^7 m/s over a distance of 2.5 cm. What is the power delivered to the electron at the instant that its displacement is 1.0 cm?

80. Coal is lifted out of a mine a vertical distance of 50 m by an engine that supplies 500 W to a conveyer belt. How much coal per minute can be brought to the surface? Ignore the effects of friction.

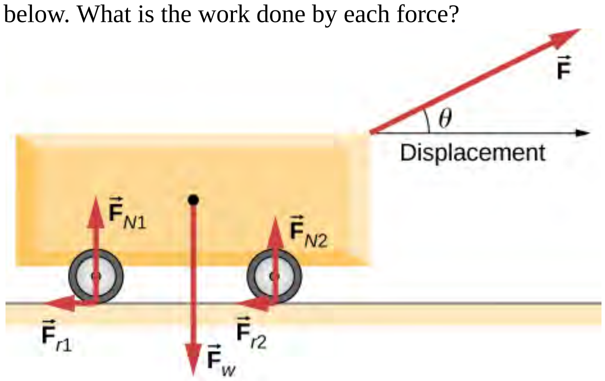
81. A girl pulls her 15-kg wagon along a flat sidewalk by applying a 10-N force at $37°$ to the horizontal. Assume that friction is negligible and that the wagon starts from rest. (a) How much work does the girl do on the wagon in the first 2.0 s. (b) How much instantaneous power does she exert at $t = 2.0$ s?

82. A typical automobile engine has an efficiency of 25%. Suppose that the engine of a 1000-kg automobile has a maximum power output of 140 hp. What is the maximum grade that the automobile can climb at 50 km/h if the frictional retarding force on it is 300 N?

83. When jogging at 13 km/h on a level surface, a 70-kg man uses energy at a rate of approximately 850 W. Using the facts that the "human engine" is approximately 25% efficient, determine the rate at which this man uses energy when jogging up a $5.0°$ slope at this same speed. Assume that the frictional retarding force is the same in both cases.

ADDITIONAL PROBLEMS

84. A cart is pulled a distance D on a flat, horizontal surface by a constant force F that acts at an angle θ with the horizontal direction. The other forces on the object during this time are gravity (F_w), normal forces (F_{N1}) and (F_{N2}), and rolling frictions F_{r1} and F_{r2}, as shown below. What is the work done by each force?



85. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1 = (3\text{ N})\hat{i} + (4\text{ N})\hat{j}$. As a result, the particle moves along the x-axis from $x = 0$ to $x = 5$ m in some time interval. What is the work done by $\vec{F}_1$?

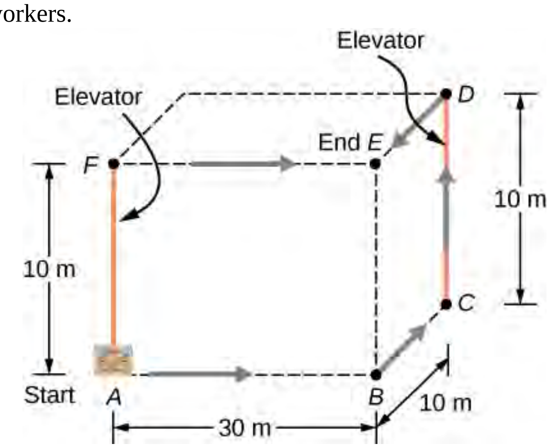
86. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1 = (3\text{ N})\hat{i} + (4\text{ N})\hat{j}$. As a result, the particle moves first along the x-axis from $x = 0$ to $x = 5$ m and then parallel to the y-axis from $y = 0$ to $y = 6$ m. What is the work done by $\vec{F}_1$?

87. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1 = (3\text{ N})\hat{i} + (4\text{ N})\hat{j}$. As a result, the particle moves along a straight path from a Cartesian coordinate of (0 m, 0 m) to (5 m, 6 m). What is the work done by $\vec{F}_1$?

88. Consider a particle on which a force acts that depends on the position of the particle. This force is given by $\vec{F}_1 = (2y)\hat{i} + (3x)\hat{j}$. Find the work done by this force when the particle moves from the origin to a point 5 meters to the right on the x-axis.

89. A boy pulls a 5-kg cart with a 20-N force at an angle of $30°$ above the horizontal for a length of time. Over this time frame, the cart moves a distance of 12 m on the horizontal floor. (a) Find the work done on the cart by the boy. (b) What will be the work done by the boy if he pulled with the same force horizontally instead of at an angle of $30°$ above the horizontal over the same distance?

90. A crate of mass 200 kg is to be brought from a site on the ground floor to a third floor apartment. The workers know that they can either use the elevator first, then slide it along the third floor to the apartment, or first slide the crate to another location marked C below, and then take the elevator to the third floor and slide it on the third floor a shorter distance. The trouble is that the third floor is very rough compared to the ground floor. Given that the coefficient of kinetic friction between the crate and the ground floor is 0.100 and between the crate and the third floor surface is 0.300, find the work needed by the workers for each path shown from A to E. Assume that the force the workers need to do is just enough to slide the crate at constant velocity (zero acceleration). *Note:* The work by the elevator against the force of gravity is not done by the workers.



91. A hockey puck of mass 0.17 kg is shot across a rough floor with the roughness different at different places, which can be described by a position-dependent coefficient of kinetic friction. For a puck moving along the x-axis, the coefficient of kinetic friction is the following function of x, where x is in m: $\mu(x) = 0.1 + 0.05x$. Find the work done by the kinetic frictional force on the hockey puck when it has moved (a) from $x = 0$ to $x = 2\,\text{m}$, and (b) from $x = 2\,\text{m}$ to $x = 4\,\text{m}$.

92. A horizontal force of 20 N is required to keep a 5.0 kg box traveling at a constant speed up a frictionless incline for a vertical height change of 3.0 m. (a) What is the work done by gravity during this change in height? (b) What is the work done by the normal force? (c) What is the work done by the horizontal force?

93. A 7.0-kg box slides along a horizontal frictionless floor at 1.7 m/s and collides with a relatively massless spring that compresses 23 cm before the box comes to a stop. (a) How much kinetic energy does the box have before it collides with the spring? (b) Calculate the work done by the spring. (c) Determine the spring constant of the spring.

94. You are driving your car on a straight road with a coefficient of friction between the tires and the road of 0.55. A large piece of debris falls in front of your view and you immediate slam on the brakes, leaving a skid mark of 30.5 m (100-feet) long before coming to a stop. A policeman sees your car stopped on the road, looks at the skid mark, and gives you a ticket for traveling over the 13.4 m/s (30 mph) speed limit. Should you fight the speeding ticket in court?

95. A crate is being pushed across a rough floor surface. If no force is applied on the crate, the crate will slow down and come to a stop. If the crate of mass 50 kg moving at speed 8 m/s comes to rest in 10 seconds, what is the rate at which the frictional force on the crate takes energy away from the crate?

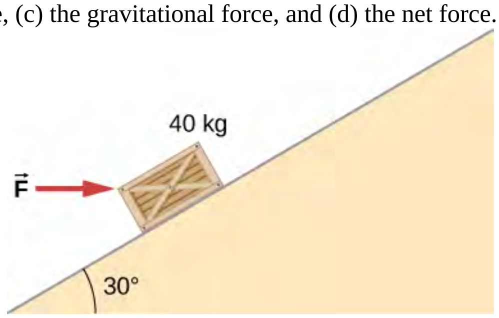
96. Suppose a horizontal force of 20 N is required to maintain a speed of 8 m/s of a 50 kg crate. (a) What is the power of this force? (b) Note that the acceleration of the crate is zero despite the fact that 20 N force acts on the crate horizontally. What happens to the energy given to the crate as a result of the work done by this 20 N force?

97. Grains from a hopper falls at a rate of 10 kg/s vertically onto a conveyor belt that is moving horizontally at a constant speed of 2 m/s. (a) What force is needed to keep the conveyor belt moving at the constant velocity? (b) What is the minimum power of the motor driving the conveyor belt?

98. A cyclist in a race must climb a $5°$ hill at a speed of 8 m/s. If the mass of the bike and the biker together is 80 kg, what must be the power output of the biker to achieve the goal?

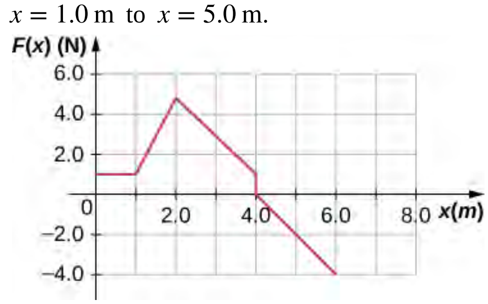
CHALLENGE PROBLEMS

99. Shown below is a 40-kg crate that is pushed at constant velocity a distance 8.0 m along a $30°$ incline by the horizontal force $\vec{\mathbf{F}}$. The coefficient of kinetic friction between the crate and the incline is $\mu_k = 0.40$. Calculate the work done by (a) the applied force, (b) the frictional force, (c) the gravitational force, and (d) the net force.



100. The surface of the preceding problem is modified so that the coefficient of kinetic friction is decreased. The same horizontal force is applied to the crate, and after being pushed 8.0 m, its speed is 5.0 m/s. How much work is now done by the force of friction? Assume that the crate starts at rest.

101. The force $F(x)$ varies with position, as shown below. Find the work done by this force on a particle as it moves from $x = 1.0\,\text{m}$ to $x = 5.0\,\text{m}$.



102. Find the work done by the same force in **Example 7.4**, between the same points, $A = (0,\ 0)$ and $B = (2\ \text{m},\ 2\ \text{m})$, over a circular arc of radius 2 m, centered at (0, 2 m). Evaluate the path integral using Cartesian coordinates. (*Hint:* You will probably need to consult a table of integrals.)

103. Answer the preceding problem using polar coordinates.

104. Find the work done by the same force in **Example 7.4**, between the same points, $A = (0,\ 0)$ and $B = (2\text{ m},\ 2\text{ m})$, over a circular arc of radius 2 m, centered at (2 m, 0). Evaluate the path integral using Cartesian coordinates. (*Hint:* You will probably need to consult a table of integrals.)

105. Answer the preceding problem using polar coordinates.

106. Constant power *P* is delivered to a car of mass *m* by its engine. Show that if air resistance can be ignored, the distance covered in a time *t* by the car, starting from rest, is given by $s = (8P/9m)^{1/2}t^{3/2}$.

107. Suppose that the air resistance a car encounters is independent of its speed. When the car travels at 15 m/s, its engine delivers 20 hp to its wheels. (a) What is the power delivered to the wheels when the car travels at 30 m/s? (b) How much energy does the car use in covering 10 km at 15 m/s? At 30 m/s? Assume that the engine is 25% efficient. (c) Answer the same questions if the force of air resistance is proportional to the speed of the automobile. (d) What do these results, plus your experience with gasoline consumption, tell you about air resistance?

108. Consider a linear spring, as in **Figure 7.7**(a), with mass *M* uniformly distributed along its length. The left end of the spring is fixed, but the right end, at the equilibrium position $x = 0,$ is moving with speed *v* in the *x*-direction. What is the total kinetic energy of the spring? (*Hint:* First express the kinetic energy of an infinitesimal element of the spring *dm* in terms of the total mass, equilibrium length, speed of the right-hand end, and position along the spring; then integrate.)

8 | POTENTIAL ENERGY AND CONSERVATION OF ENERGY

Figure 8.1 Shown here is part of a Ball Machine sculpture by George Rhoads. A ball in this contraption is lifted, rolls, falls, bounces, and collides with various objects, but throughout its travels, its kinetic energy changes in definite, predictable amounts, which depend on its position and the objects with which it interacts. (credit: modification of work by Roland Tanglao)

Chapter Outline

Introduction

In George Rhoads' rolling ball sculpture, the principle of conservation of energy governs the changes in the ball's kinetic energy and relates them to changes and transfers for other types of energy associated with the ball's interactions. In this chapter, we introduce the important concept of potential energy. This will enable us to formulate the law of conservation of mechanical energy and to apply it to simple systems, making solving problems easier. In the final section on sources of energy, we will consider energy transfers and the general law of conservation of energy. Throughout this book, the law of conservation of energy will be applied in increasingly more detail, as you encounter more complex and varied systems, and other forms of energy.

8.1 | Potential Energy of a System

Learning Objectives

By the end of this section, you will be able to:

- Relate the difference of potential energy to work done on a particle for a system without friction or air drag
- Explain the meaning of the zero of the potential energy function for a system
- Calculate and apply the gravitational potential energy for an object near Earth's surface and the elastic potential energy of a mass-spring system

In **Work**, we saw that the work done on an object by the constant gravitational force, near the surface of Earth, over any displacement is a function only of the difference in the positions of the end-points of the displacement. This property allows us to define a different kind of energy for the system than its kinetic energy, which is called **potential energy**. We consider various properties and types of potential energy in the following subsections.

Potential Energy Basics

In **Motion in Two and Three Dimensions**, we analyzed the motion of a projectile, like kicking a football in **Figure 8.2**. For this example, let's ignore friction and air resistance. As the football rises, the work done by the gravitational force on the football is negative, because the ball's displacement is positive vertically and the force due to gravity is negative vertically. We also noted that the ball slowed down until it reached its highest point in the motion, thereby decreasing the ball's kinetic energy. This loss in kinetic energy translates to a gain in gravitational potential energy of the football-Earth system.

As the football falls toward Earth, the work done on the football is now positive, because the displacement and the gravitational force both point vertically downward. The ball also speeds up, which indicates an increase in kinetic energy. Therefore, energy is converted from gravitational potential energy back into kinetic energy.

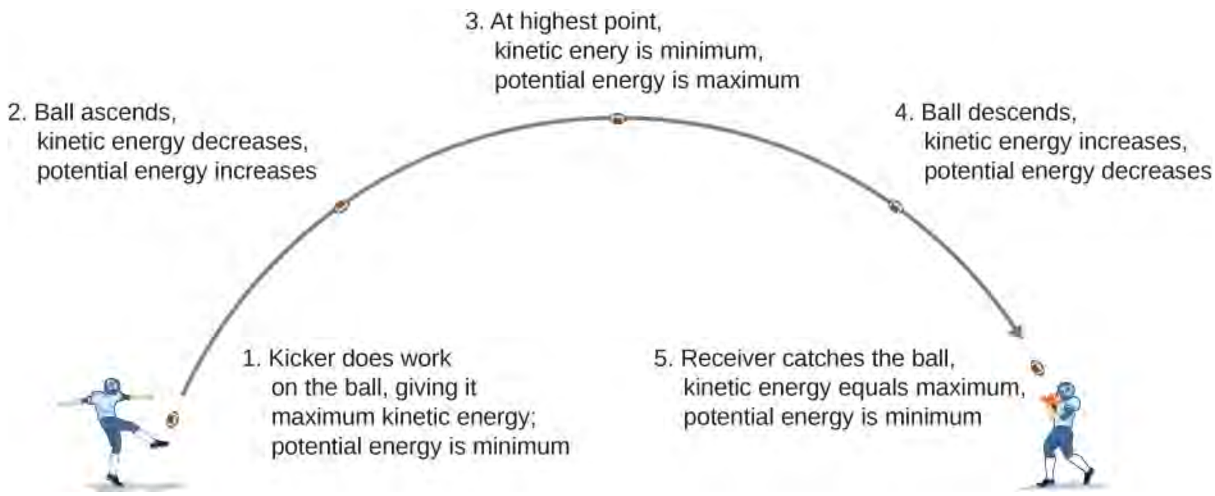


Figure 8.2 As a football starts its descent toward the wide receiver, gravitational potential energy is converted back into kinetic energy.

Based on this scenario, we can define the difference of potential energy from point A to point B as the negative of the work done:

$$\Delta U_{AB} = U_B - U_A = -W_{AB}. \tag{8.1}$$

This formula explicitly states a **potential energy difference**, not just an absolute potential energy. Therefore, we need to define potential energy at a given position in such a way as to state standard values of potential energy on their own, rather than potential energy differences. We do this by rewriting the potential energy function in terms of an arbitrary constant,

$$\Delta U = U(\vec{\mathbf{r}}) - U(\vec{\mathbf{r}}_0). \tag{8.2}$$

The choice of the potential energy at a starting location of $\vec{\mathbf{r}}_0$ is made out of convenience in the given problem. Most importantly, whatever choice is made should be stated and kept consistent throughout the given problem. There are some well-accepted choices of initial potential energy. For example, the lowest height in a problem is usually defined as zero potential energy, or if an object is in space, the farthest point away from the system is often defined as zero potential energy. Then, the potential energy, with respect to zero at $\vec{\mathbf{r}}_0$, is just $U(\vec{r})$.

As long as there is no friction or air resistance, the change in kinetic energy of the football equals the change in gravitational potential energy of the football. This can be generalized to any potential energy:

$$\Delta K_{AB} = \Delta U_{AB}. \tag{8.3}$$

Let's look at a specific example, choosing zero potential energy for gravitational potential energy at convenient points.

Example 8.1

Basic Properties of Potential Energy

A particle moves along the *x*-axis under the action of a force given by $F = -ax^2$, where $a = 3\ \text{N/m}^2$. (a) What is the difference in its potential energy as it moves from $x_A = 1\ \text{m}$ to $x_B = 2\ \text{m}$? (b) What is the particle's potential energy at $x = 1\ \text{m}$ with respect to a given 0.5 J of potential energy at $x = 0$?

Strategy

(a) The difference in potential energy is the negative of the work done, as defined by **Equation 8.1**. The work is defined in the previous chapter as the dot product of the force with the distance. Since the particle is moving forward in the *x*-direction, the dot product simplifies to a multiplication ($\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = 1$). To find the total work done, we need to integrate the function between the given limits. After integration, we can state the work or the potential energy. (b) The potential energy function, with respect to zero at $x = 0$, is the indefinite integral encountered in part (a), with the constant of integration determined from **Equation 8.3**. Then, we substitute the *x*-value into the function of potential energy to calculate the potential energy at $x = 1$ m.

Solution

a. The work done by the given force as the particle moves from coordinate *x* to $x + dx$ in one dimension is

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = F dx = -ax^2 dx.$$

Substituting this expression into **Equation 8.1**, we obtain

$$\Delta U = -W = \int_{x_1}^{x_2} ax^2\, dx = \frac{1}{3}(3\ \text{N/m}^2)x^2\Big|_{1\ \text{m}}^{2\ \text{m}} = 7\ \text{J}.$$

b. The indefinite integral for the potential energy function in part (a) is

$$U(x) = \frac{1}{3}ax^3 + \text{const.},$$

and we want the constant to be determined by

$$U(0) = 0.5\ \text{J}.$$

Thus, the potential energy with respect to zero at $x = 0$ is just

$$U(x) = \frac{1}{3}ax^3 + 0.5\text{ J}.$$

Therefore, the potential energy at $x = 1\text{ m}$ is

$$U(1\text{ m}) = \frac{1}{3}\left(3\text{ N/m}^2\right)(1\text{ m})^3 + 0.5\text{ J} = 1.5\text{ J}.$$

Significance

In this one-dimensional example, any function we can integrate, independent of path, is conservative. Notice how we applied the definition of potential energy difference to determine the potential energy function with respect to zero at a chosen point. Also notice that the potential energy, as determined in part (b), at $x = 1\text{ m}$ is $U(1\text{ m}) = 1\text{ J}$ and at $x = 2\text{ m}$ is $U(2\text{ m}) = 8\text{ J}$; their difference is the result in part (a).

8.1 Check Your Understanding In **Example 8.1**, what are the potential energies of the particle at $x = 1\text{ m}$ and $x = 2\text{ m}$ with respect to zero at $x = 1.5\text{ m}$? Verify that the difference of potential energy is still 7 J.

Systems of Several Particles

In general, a system of interest could consist of several particles. The difference in the potential energy of the system is the negative of the work done by gravitational or elastic forces, which, as we will see in the next section, are conservative forces. The potential energy difference depends only on the initial and final positions of the particles, and on some parameters that characterize the interaction (like mass for gravity or the spring constant for a Hooke's law force).

It is important to remember that potential energy is a property of the interactions between objects in a chosen system, and not just a property of each object. This is especially true for electric forces, although in the examples of potential energy we consider below, parts of the system are either so big (like Earth, compared to an object on its surface) or so small (like a massless spring), that the changes those parts undergo are negligible if included in the system.

Types of Potential Energy

For each type of interaction present in a system, you can label a corresponding type of potential energy. The total potential energy of the system is the sum of the potential energies of all the types. (This follows from the additive property of the dot product in the expression for the work done.) Let's look at some specific examples of types of potential energy discussed in **Work**. First, we consider each of these forces when acting separately, and then when both act together.

Gravitational potential energy near Earth's surface

The system of interest consists of our planet, Earth, and one or more particles near its surface (or bodies small enough to be considered as particles, compared to Earth). The gravitational force on each particle (or body) is just its weight *mg* near the surface of Earth, acting vertically down. According to Newton's third law, each particle exerts a force on Earth of equal magnitude but in the opposite direction. Newton's second law tells us that the magnitude of the acceleration produced by each of these forces on Earth is *mg* divided by Earth's mass. Since the ratio of the mass of any ordinary object to the mass of Earth is vanishingly small, the motion of Earth can be completely neglected. Therefore, we consider this system to be a group of single-particle systems, subject to the uniform gravitational force of Earth.

In **Work**, the work done on a body by Earth's uniform gravitational force, near its surface, depended on the mass of the body, the acceleration due to gravity, and the difference in height the body traversed, as given by **Equation 7.4**. By definition, this work is the negative of the difference in the gravitational potential energy, so that difference is

$$\Delta U_{\text{grav}} = -W_{\text{grav, }AB} = mg(y_B - y_A). \quad \textbf{(8.4)}$$

You can see from this that the gravitational potential energy function, near Earth's surface, is

$$U(y) = mgy + \text{const.} \quad \textbf{(8.5)}$$

You can choose the value of the constant, as described in the discussion of **Equation 8.2**; however, for solving most problems, the most convenient constant to choose is zero for when $y = 0,$ which is the lowest vertical position in the problem.

Figure 8.3 Don't jump—you have so much potential (gravitational potential energy, that is). (credit: Andy Spearing)

Example 8.2

Gravitational Potential Energy of a Hiker

The summit of Great Blue Hill in Milton, MA, is 147 m above its base and has an elevation above sea level of 195 m (**Figure 8.4**). (Its Native American name, *Massachusett*, was adopted by settlers for naming the Bay Colony and state near its location.) A 75-kg hiker ascends from the base to the summit. What is the gravitational potential energy of the hiker-Earth system with respect to zero gravitational potential energy at base height, when the hiker is (a) at the base of the hill, (b) at the summit, and (c) at sea level, afterward?

Figure 8.4 Sketch of the profile of Great Blue Hill, Milton, MA. The altitudes of the three levels are indicated.

Strategy

First, we need to pick an origin for the *y*-axis and then determine the value of the constant that makes the potential energy zero at the height of the base. Then, we can determine the potential energies from **Equation 8.5**, based on the relationship between the zero potential energy height and the height at which the hiker is located.

Solution

a. Let's choose the origin for the *y*-axis at base height, where we also want the zero of potential energy to be. This choice makes the constant equal to zero and

$$U(\text{base}) = U(0) = 0.$$

b. At the summit, $y = 147\text{ m}$, so

$$U(\text{summit}) = U(147\text{ m}) = mgh = (75 \times 9.8\text{ N})(147\text{ m}) = 108\text{ kJ}.$$

c. At sea level, $y = (147 - 195)\text{m} = -48\text{ m}$, so

$$U(\text{sea-level}) = (75 \times 9.8\text{ N})(-48\text{ m}) = -35.3\text{ kJ}.$$

Significance

Besides illustrating the use of **Equation 8.4** and **Equation 8.5**, the values of gravitational potential energy we found are reasonable. The gravitational potential energy is higher at the summit than at the base, and lower at sea level than at the base. Gravity does work on you on your way up, too! It does negative work and not quite as much (in magnitude), as your muscles do. But it certainly does work. Similarly, your muscles do work on your way down, as negative work. The numerical values of the potential energies depend on the choice of zero of potential energy, but the physically meaningful differences of potential energy do not. [Note that since **Equation 8.2** is a difference, the numerical values do not depend on the origin of coordinates.]

8.2 Check Your Understanding What are the values of the gravitational potential energy of the hiker at the base, summit, and sea level, with respect to a sea-level zero of potential energy?

Elastic potential energy

In **Work**, we saw that the work done by a perfectly elastic spring, in one dimension, depends only on the spring constant and the squares of the displacements from the unstretched position, as given in **Equation 7.5**. This work involves only the properties of a Hooke's law interaction and not the properties of real springs and whatever objects are attached to them. Therefore, we can define the difference of elastic potential energy for a spring force as the negative of the work done by the spring force in this equation, before we consider systems that embody this type of force. Thus,

$$\Delta U = -W_{AB} = \frac{1}{2}k(x_B^2 - x_A^2), \tag{8.6}$$

where the object travels from point A to point B. The potential energy function corresponding to this difference is

$$U(x) = \frac{1}{2}kx^2 + \text{const.} \tag{8.7}$$

If the spring force is the only force acting, it is simplest to take the zero of potential energy at $x = 0$, when the spring is at its unstretched length. Then, the constant is **Equation 8.7** is zero. (Other choices may be more convenient if other forces are acting.)

Example 8.3

Spring Potential Energy

A system contains a perfectly elastic spring, with an unstretched length of 20 cm and a spring constant of 4 N/cm. (a) How much elastic potential energy does the spring contribute when its length is 23 cm? (b) How much more potential energy does it contribute if its length increases to 26 cm?

Strategy

When the spring is at its unstretched length, it contributes nothing to the potential energy of the system, so we can use **Equation 8.7** with the constant equal to zero. The value of x is the length minus the unstretched length. When the spring is expanded, the spring's displacement or difference between its relaxed length and stretched length should be used for the x-value in calculating the potential energy of the spring.

Solution

a. The displacement of the spring is $x = 23\text{ cm} - 20\text{ cm} = 3\text{ cm}$, so the contributed potential energy is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(4\ \text{N/cm})(3\ \text{cm})^2 = 0.18\ \text{J}.$$

b. When the spring's displacement is $x = 26\ \text{cm} - 20\ \text{cm} = 6\ \text{cm}$, the potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}(4\ \text{N/cm})(6\ \text{cm})^2 = 0.72\ \text{J}$, which is a 0.54-J increase over the amount in part (a).

Significance

Calculating the elastic potential energy and potential energy differences from **Equation 8.7** involves solving for the potential energies based on the given lengths of the spring. Since U depends on x^2, the potential energy for a compression (negative x) is the same as for an extension of equal magnitude.

8.3 Check Your Understanding When the length of the spring in **Example 8.3** changes from an initial value of 22.0 cm to a final value, the elastic potential energy it contributes changes by $-0.0800\ \text{J}$. Find the final length.

Gravitational and elastic potential energy

A simple system embodying both gravitational and elastic types of potential energy is a one-dimensional, vertical mass-spring system. This consists of a massive particle (or block), hung from one end of a perfectly elastic, massless spring, the other end of which is fixed, as illustrated in **Figure 8.5**.

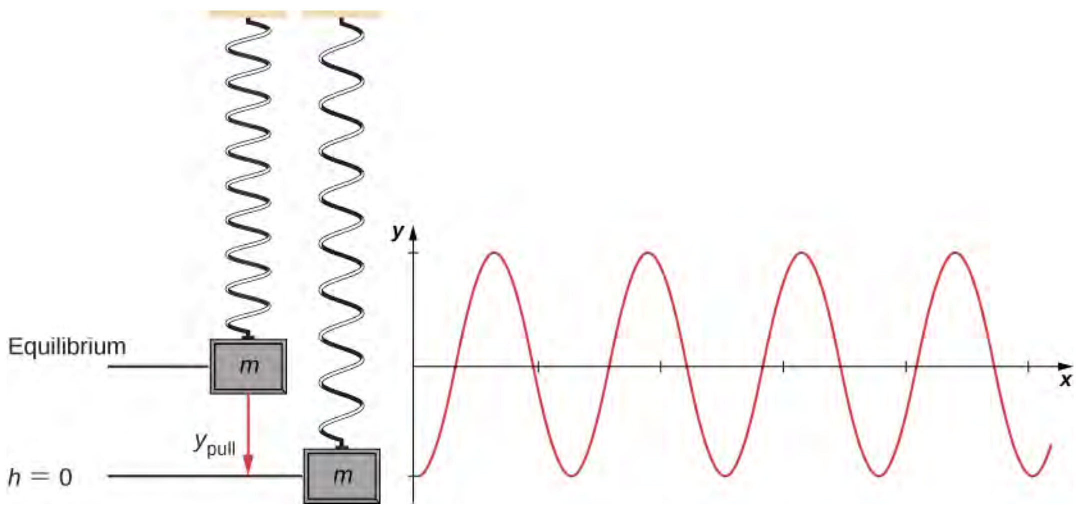


Figure 8.5 A vertical mass-spring system, with the y-axis pointing upwards. The mass is initially at an equilibrium position and pulled downward to y_{pull}. An oscillation begins, centered at the equilibrium position.

First, let's consider the potential energy of the system. Assuming the spring is massless, the system of the block and Earth gains and loses potential energy. We need to define the constant in the potential energy function of **Equation 8.5**. Often, the ground is a suitable choice for when the gravitational potential energy is zero; however, in this case, the lowest point or when $h = 0$ is a convenient location for zero gravitational potential energy. Note that this choice is arbitrary, and the problem can be solved correctly even if another choice is picked.

We must also define the elastic potential energy of the system and the corresponding constant, as detailed in **Equation 8.7**. The equilibrium location is the most suitable mathematically to choose for where the potential energy of the spring is zero.

Therefore, based on this convention, each potential energy and kinetic energy can be written out for three critical points of the system: (1) the lowest pulled point, (2) the equilibrium position of the spring, and (3) the highest point achieved. We note that the total energy of the system is conserved, so any total energy in this chart could be matched up to solve for an unknown quantity. The results are shown in **Table 8.1**.

	Gravitational P.E.	Elastic P.E.	Kinetic E.
(3) Highest Point	$2mgy_{\text{pull}}$	$\frac{1}{2}ky^2{}_{\text{pull}}$	0
(2) Equilibrium	mgy_{pull}	0	$\frac{1}{2}mv^2$
(1) Lowest Point	0	$\frac{1}{2}ky^2{}_{\text{pull}}$	0

Table 8.1 Components of Energy in a Vertical Mass-Spring System

Figure 8.6 A bungee jumper transforms gravitational potential energy at the start of the jump into elastic potential energy at the bottom of the jump.

Example 8.4

Potential Energy of a Vertical Mass-Spring System

A block weighing $12\,\text{N}$ is hung from a spring with a spring constant of $6.0\,\text{N/m}$, as shown in **Figure 8.5**. The block is pulled down an additional $5.0\,\text{cm}$ from its equilibrium position and released. (a) What is the difference in just the spring potential energy, from an initial equilibrium position to its pulled-down position? (b) What is the difference in just the gravitational potential energy from its initial equilibrium position to its pulled-down position? (c) What is the kinetic energy of the block as it passes through the equilibrium position from its pulled-down position?

Strategy

In parts (a) and (b), we want to find a difference in potential energy, so we can use **Equation 8.6** and **Equation 8.4**, respectively. Each of these expressions takes into consideration the change in the energy relative to another position, further emphasizing that potential energy is calculated with a reference or second point in mind. By choosing the conventions of the lowest point in the diagram where the gravitational potential energy is zero and the equilibrium position of the spring where the elastic potential energy is zero, these differences in energies can now be calculated. In part (c), we take a look at the differences between the two potential energies. The difference between the two results in kinetic energy, since there is no friction or drag in this system that can take energy from the system.

Solution

a. Since the gravitational potential energy is zero at the lowest point, the change in gravitational potential energy is

$$\Delta U_{\text{grav}} = mgy - 0 = (12\,\text{N})(5.0\,\text{cm}) = 0.60\,\text{J}.$$

b. The equilibrium position of the spring is defined as zero potential energy. Therefore, the change in elastic potential energy is

$$\Delta U_{\text{elastic}} = 0 - \frac{1}{2}ky_{\text{pull}}^2 = -\left(\frac{1}{2}\right)\left(6.0\frac{\text{N}}{\text{m}}\right)(5.0\,\text{cm})^2 = -0.75\,\text{J}.$$

c. The block started off being pulled downward with a relative potential energy of $0.75\,\text{J}$. The gravitational potential energy required to rise $5.0\,\text{cm}$ is $0.60\,\text{J}$. The energy remaining at this equilibrium position must be kinetic energy. We can solve for this gain in kinetic energy from **Equation 8.2**,

$$\Delta K = -(\Delta U_{\text{elastic}} + \Delta U_{\text{grav}}) = -(-0.75\,\text{J} + 0.60\,\text{J}) = 0.15\,\text{J}.$$

Significance

Even though the potential energies are relative to a chosen zero location, the solutions to this problem would be the same if the zero energy points were chosen at different locations.

8.4 Check Your Understanding Suppose the mass in **Example 8.4** is in equilibrium, and you pull it down another 3.0 cm, making the pulled-down distance a total of $8.0\,\text{cm}$. The elastic potential energy of the spring increases, because you're stretching it more, but the gravitational potential energy of the mass decreases, because you're lowering it. Does the total potential energy increase, decrease, or remain the same?

View this **simulation (https://openstaxcollege.org/l/21conenerskat)** to learn about conservation of energy with a skater! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

A sample chart of a variety of energies is shown in **Table 8.2** to give you an idea about typical energy values associated with certain events. Some of these are calculated using kinetic energy, whereas others are calculated by using quantities found in a form of potential energy that may not have been discussed at this point.

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Annual world energy use	4.0×10^{20}
Large fusion bomb (9 megaton)	3.8×10^{16}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}
1 barrel crude oil	5.9×10^{9}
1 ton TNT	4.2×10^{9}
1 gallon of gasoline	1.2×10^{8}
Daily adult food intake (recommended)	1.2×10^{7}
1000-kg car at 90 km/h	3.1×10^{5}
Tennis ball at 100 km/h	22

Table 8.2 Energy of Various Objects and Phenomena

Object/phenomenon	Energy in joules
Mosquito $\left(10^{-2}\text{ g at }0.5\text{ m/s}\right)$	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

Table 8.2 Energy of Various Objects and Phenomena

8.2 | Conservative and Non-Conservative Forces

Learning Objectives

By the end of this section, you will be able to:

- Characterize a conservative force in several different ways
- Specify mathematical conditions that must be satisfied by a conservative force and its components
- Relate the conservative force between particles of a system to the potential energy of the system
- Calculate the components of a conservative force in various cases

In **Potential Energy and Conservation of Energy**, any transition between kinetic and potential energy conserved the total energy of the system. This was path independent, meaning that we can start and stop at any two points in the problem, and the total energy of the system—kinetic plus potential—at these points are equal to each other. This is characteristic of a **conservative force**. We dealt with conservative forces in the preceding section, such as the gravitational force and spring force. When comparing the motion of the football in **Figure 8.2**, the total energy of the system never changes, even though the gravitational potential energy of the football increases, as the ball rises relative to ground and falls back to the initial gravitational potential energy when the football player catches the ball. **Non-conservative forces** are dissipative forces such as friction or air resistance. These forces take energy away from the system as the system progresses, energy that you can't get back. These forces are path dependent; therefore it matters where the object starts and stops.

Conservative Force

The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points:

$$W_{AB,\text{ path-1}} = \int_{AB,\text{ path-1}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} = W_{AB,\text{ path-2}} = \int_{AB,\text{ path-2}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}}. \tag{8.8}$$

The work done by a non-conservative force depends on the path taken.

Equivalently, a force is conservative if the work it does around any closed path is zero:

$$W_{\text{closed path}} = \oint \vec{\mathbf{E}}_{\text{cons}} \cdot d\vec{\mathbf{r}} = 0. \tag{8.9}$$

[In **Equation 8.9**, we use the notation of a circle in the middle of the integral sign for a line integral over a closed path, a notation found in most physics and engineering texts.] **Equation 8.8** and **Equation 8.9** are equivalent because any closed path is the sum of two paths: the first going from A to B, and the second going from B to A. The work done going along a path from B to A is the negative of the work done going along the same path from A to B, where A and B are any two points on the closed path:

$$0 = \oint \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} = \int_{AB,\ \text{path-1}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} + \int_{BA,\ \text{path-2}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}}$$
$$= \int_{AB,\ \text{path-1}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} - \int_{AB,\ \text{path-2}} \vec{\mathbf{F}}_{\text{cons}} \cdot d\vec{\mathbf{r}} = 0.$$

You might ask how we go about proving whether or not a force is conservative, since the definitions involve any and all paths from A to B, or any and all closed paths, but to do the integral for the work, you have to choose a particular path. One answer is that the work done is independent of path if the infinitesimal work $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is an **exact differential**, the way the infinitesimal net work was equal to the exact differential of the kinetic energy, $dW_{\text{net}} = m\vec{\mathbf{v}} \cdot d\vec{\mathbf{v}} = d\frac{1}{2}mv^2$, when we derived the work-energy theorem in **Work-Energy Theorem**. There are mathematical conditions that you can use to test whether the infinitesimal work done by a force is an exact differential, and the force is conservative. These conditions only involve differentiation and are thus relatively easy to apply. In two dimensions, the condition for $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = F_x dx + F_y dy$ to be an exact differential is

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}. \tag{8.10}$$

You may recall that the work done by the force in **Example 7.4** depended on the path. For that force,

$$F_x = (5\ \text{N/m})y \text{ and } F_y = (10\ \text{N/m})x.$$

Therefore,

$$(dF_x/dy) = 5\ \text{N/m} \neq (dF_y/dx) = 10\ \text{N/m},$$

which indicates it is a non-conservative force. Can you see what you could change to make it a conservative force?

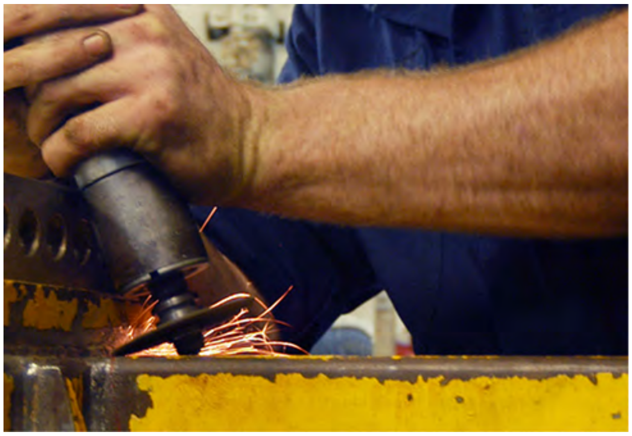


Figure 8.7 A grinding wheel applies a non-conservative force, because the work done depends on how many rotations the wheel makes, so it is path-dependent.

Example 8.5

Conservative or Not?

Which of the following two-dimensional forces are conservative and which are not? Assume a and b are constants with appropriate units:

(a) $axy^3\,\hat{\mathbf{i}} + ayx^3\,\hat{\mathbf{j}}$, (b) $a\left[\left(y^2/x\right)\hat{\mathbf{i}} + 2y\ln(x/b)\,\hat{\mathbf{j}}\right]$, (c) $\dfrac{ax\,\hat{\mathbf{i}} + ay\,\hat{\mathbf{j}}}{x^2+y^2}$

Strategy

Apply the condition stated in **Equation 8.10**, namely, using the derivatives of the components of each force indicated. If the derivative of the *y*-component of the force with respect to *x* is equal to the derivative of the *x*-component of the force with respect to *y*, the force is a conservative force, which means the path taken for potential energy or work calculations always yields the same results.

Solution

a. $\dfrac{dF_x}{dy} = \dfrac{d\left(axy^3\right)}{dy} = 3axy^2$ and $\dfrac{dF_y}{dx} = \dfrac{d\left(ayx^3\right)}{dx} = 3ayx^2$, so this force is non-conservative.

b. $\dfrac{dF_x}{dy} = \dfrac{d\left(ay^2/x\right)}{dy} = \dfrac{2ay}{x}$ and $\dfrac{dF_y}{dx} = \dfrac{d(2ay\ln(x/b))}{dx} = \dfrac{2ay}{x}$, so this force is conservative.

c. $\dfrac{dF_x}{dy} = \dfrac{d\left(ax/\left(x^2+y^2\right)\right)}{dy} = -\dfrac{ax(2y)}{\left(x^2+y^2\right)^2} = \dfrac{dF_y}{dx} = \dfrac{d\left(ay/\left(x^2+y^2\right)\right)}{dx}$, again conservative.

Significance

The conditions in **Equation 8.10** are derivatives as functions of a single variable; in three dimensions, similar conditions exist that involve more derivatives.

8.5 Check Your Understanding A two-dimensional, conservative force is zero on the *x*- and *y*-axes, and satisfies the condition $(dF_x/dy) = \left(dF_y/dx\right) = \left(4\text{ N/m}^3\right)xy$. What is the magnitude of the force at the point $x = y = 1\text{ m}$?

Before leaving this section, we note that non-conservative forces do not have potential energy associated with them because the energy is lost to the system and can't be turned into useful work later. So there is always a conservative force associated with every potential energy. We have seen that potential energy is defined in relation to the work done by conservative forces. That relation, **Equation 8.1**, involved an integral for the work; starting with the force and displacement, you integrated to get the work and the change in potential energy. However, integration is the inverse operation of differentiation; you could equally well have started with the potential energy and taken its derivative, with respect to displacement, to get the force. The infinitesimal increment of potential energy is the dot product of the force and the infinitesimal displacement,

$$dU = -\vec{\mathbf{F}}\cdot d\vec{\mathbf{l}} = -F_l\,dl.$$

Here, we chose to represent the displacement in an arbitrary direction by $d\vec{\mathbf{l}}$, so as not to be restricted to any particular coordinate direction. We also expressed the dot product in terms of the magnitude of the infinitesimal displacement and the component of the force in its direction. Both these quantities are scalars, so you can divide by *dl* to get

$$F_l = -\frac{dU}{dl}. \qquad (8.11)$$

This equation gives the relation between force and the potential energy associated with it. In words, the component of a conservative force, in a particular direction, equals the negative of the derivative of the corresponding potential energy, with respect to a displacement in that direction. For one-dimensional motion, say along the *x*-axis, **Equation 8.11** give the entire vector force, $\bar{\mathbf{F}} = F_x\,\hat{\mathbf{i}} = -\dfrac{\partial U}{\partial x}\hat{\mathbf{i}}$.

In two dimensions,

$$\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} = -\left(\frac{\partial U}{\partial x}\right)\hat{\mathbf{i}} - \left(\frac{\partial U}{\partial y}\right)\hat{\mathbf{j}}.$$

From this equation, you can see why **Equation 8.11** is the condition for the work to be an exact differential, in terms of the derivatives of the components of the force. In general, a partial derivative notation is used. If a function has many variables in it, the derivative is taken only of the variable the partial derivative specifies. The other variables are held constant. In three dimensions, you add another term for the *z*-component, and the result is that the force is the negative of the gradient of the potential energy. However, we won't be looking at three-dimensional examples just yet.

Example 8.6

Force due to a Quartic Potential Energy

The potential energy for a particle undergoing one-dimensional motion along the *x*-axis is

$$U(x) = \frac{1}{4}cx^4,$$

where $c = 8\ \text{N/m}^3$. Its total energy at $x = 0$ is $2\ \text{J}$, and it is not subject to any non-conservative forces. Find (a) the positions where its kinetic energy is zero and (b) the forces at those positions.

Strategy

(a) We can find the positions where $K = 0$, so the potential energy equals the total energy of the given system.

(b) Using **Equation 8.11**, we can find the force evaluated at the positions found from the previous part, since the mechanical energy is conserved.

Solution

a. The total energy of the system of 2 J equals the quartic elastic energy as given in the problem,

$$2\ \text{J} = \frac{1}{4}\left(8\ \text{N/m}^3\right)x_f{}^4.$$

Solving for x_f results in $x_f = \pm 1\ \text{m}$.

b. From **Equation 8.11**,

$$F_x = -dU/dx = -cx^3.$$

Thus, evaluating the force at $\pm 1\ \text{m}$, we get

$$\vec{\mathbf{F}} = -(8\ \text{N/m}^3)(\pm 1\ \text{m})^3\,\hat{\mathbf{i}} = \pm 8\ \text{N}\,\hat{\mathbf{i}}.$$

At both positions, the magnitude of the forces is 8 N and the directions are toward the origin, since this is the potential energy for a restoring force.

Significance

Finding the force from the potential energy is mathematically easier than finding the potential energy from the force, because differentiating a function is generally easier than integrating one.

8.6 Check Your Understanding Find the forces on the particle in **Example 8.6** when its kinetic energy is 1.0 J at $x = 0$.

8.3 | Conservation of Energy

Learning Objectives

By the end of this section, you will be able to:

- Formulate the principle of conservation of mechanical energy, with or without the presence of non-conservative forces
- Use the conservation of mechanical energy to calculate various properties of simple systems

In this section, we elaborate and extend the result we derived in **Potential Energy of a System**, where we re-wrote the work-energy theorem in terms of the change in the kinetic and potential energies of a particle. This will lead us to a discussion of the important principle of the conservation of mechanical energy. As you continue to examine other topics in physics, in later chapters of this book, you will see how this conservation law is generalized to encompass other types of energy and energy transfers. The last section of this chapter provides a preview.

The terms 'conserved quantity' and 'conservation law' have specific, scientific meanings in physics, which are different from the everyday meanings associated with the use of these words. (The same comment is also true about the scientific and everyday uses of the word 'work.') In everyday usage, you could conserve water by not using it, or by using less of it, or by re-using it. Water is composed of molecules consisting of two atoms of hydrogen and one of oxygen. Bring these atoms together to form a molecule and you create water; dissociate the atoms in such a molecule and you destroy water. However, in scientific usage, a **conserved quantity** for a system stays constant, changes by a definite amount that is transferred to other systems, and/or is converted into other forms of that quantity. A conserved quantity, in the scientific sense, can be transformed, but not strictly created or destroyed. Thus, there is no physical law of conservation of water.

Systems with a Single Particle or Object

We first consider a system with a single particle or object. Returning to our development of **Equation 8.2**, recall that we first separated all the forces acting on a particle into conservative and non-conservative types, and wrote the work done by each type of force as a separate term in the work-energy theorem. We then replaced the work done by the conservative forces by the change in the potential energy of the particle, combining it with the change in the particle's kinetic energy to get **Equation 8.2**. Now, we write this equation without the middle step and define the sum of the kinetic and potential energies, $K + U = E;$ to be the **mechanical energy** of the particle.

Conservation of Energy

The mechanical energy E of a particle stays constant unless forces outside the system or non-conservative forces do work on it, in which case, the change in the mechanical energy is equal to the work done by the non-conservative forces:

$$W_{\text{nc, } AB} = \Delta(K + U)_{AB} = \Delta E_{AB}. \tag{8.12}$$

This statement expresses the concept of **energy conservation** for a classical particle as long as there is no non-conservative work. Recall that a classical particle is just a point mass, is nonrelativistic, and obeys Newton's laws of motion. In **Relativity (http://cnx.org/content/m58555/latest/)** , we will see that conservation of energy still applies to a non-classical particle, but for that to happen, we have to make a slight adjustment to the definition of energy.

It is sometimes convenient to separate the case where the work done by non-conservative forces is zero, either because no such forces are assumed present, or, like the normal force, they do zero work when the motion is parallel to the surface. Then

$$0 = W_{\text{nc, } AB} = \Delta(K + U)_{AB} = \Delta E_{AB}. \tag{8.13}$$

In this case, the conservation of mechanical energy can be expressed as follows: The mechanical energy of a particle does not change if all the non-conservative forces that may act on it do no work. Understanding the concept of energy conservation is the important thing, not the particular equation you use to express it.

Problem-Solving Strategy: Conservation of Energy

1. Identify the body or bodies to be studied (the system). Often, in applications of the principle of mechanical energy conservation, we study more than one body at the same time.
2. Identify all forces acting on the body or bodies.
3. Determine whether each force that does work is conservative. If a non-conservative force (e.g., friction) is doing work, then mechanical energy is not conserved. The system must then be analyzed with non-conservative work, **Equation 8.13**.
4. For every force that does work, choose a reference point and determine the potential energy function for the force. The reference points for the various potential energies do not have to be at the same location.
5. Apply the principle of mechanical energy conservation by setting the sum of the kinetic energies and potential energies equal at every point of interest.

Example 8.7

Simple Pendulum

A particle of mass m is hung from the ceiling by a massless string of length 1.0 m, as shown in **Figure 8.8**. The particle is released from rest, when the angle between the string and the downward vertical direction is $30°$. What is its speed when it reaches the lowest point of its arc?

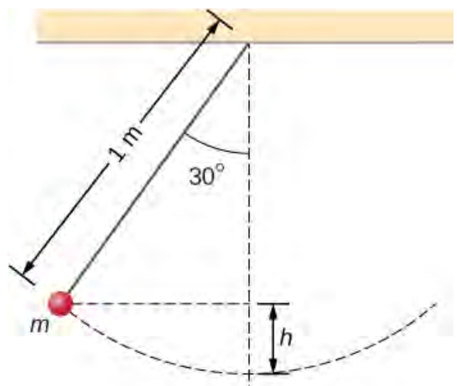


Figure 8.8 A particle hung from a string constitutes a simple pendulum. It is shown when released from rest, along with some distances used in analyzing the motion.

Strategy

Using our problem-solving strategy, the first step is to define that we are interested in the particle-Earth system. Second, only the gravitational force is acting on the particle, which is conservative (step 3). We neglect air resistance in the problem, and no work is done by the string tension, which is perpendicular to the arc of the motion. Therefore, the mechanical energy of the system is conserved, as represented by **Equation 8.13**, $0 = \Delta(K + U)$. Because the particle starts from rest, the increase in the kinetic energy is just the kinetic energy at the lowest point. This increase in kinetic energy equals the decrease in the gravitational potential energy, which we can calculate from the geometry. In step 4, we choose a reference point for zero gravitational potential energy to be at the lowest vertical point the particle achieves, which is mid-swing. Lastly, in step 5, we set the sum of energies at the highest point (initial) of the swing to the lowest point (final) of the swing to ultimately solve for the final speed.

Solution

We are neglecting non-conservative forces, so we write the energy conservation formula relating the particle at the highest point (initial) and the lowest point in the swing (final) as

$$K_{\mathrm{i}} + U_{\mathrm{i}} = K_{\mathrm{f}} + U_{\mathrm{f}}.$$

Since the particle is released from rest, the initial kinetic energy is zero. At the lowest point, we define the gravitational potential energy to be zero. Therefore our conservation of energy formula reduces to

$$\begin{aligned} 0 + mgh &= \frac{1}{2}mv^2 + 0 \\ v &= \sqrt{2gh}. \end{aligned}$$

The vertical height of the particle is not given directly in the problem. This can be solved for by using trigonometry and two givens: the length of the pendulum and the angle through which the particle is vertically pulled up. Looking at the diagram, the vertical dashed line is the length of the pendulum string. The vertical height is labeled h. The other partial length of the vertical string can be calculated with trigonometry. That piece is solved for by

$$\cos\theta = x/L,\ x = L\cos\theta.$$

Therefore, by looking at the two parts of the string, we can solve for the height h,

$$\begin{aligned} x + h &= L \\ L\cos\theta + h &= L \\ h &= L - L\cos\theta = L(1 - \cos\theta). \end{aligned}$$

We substitute this height into the previous expression solved for speed to calculate our result:

$$v = \sqrt{2gL(1-\cos\theta)} = \sqrt{2\left(9.8\ \mathrm{m/s^2}\right)(1\ \mathrm{m})(1-\cos 30°)} = 1.62\ \mathrm{m/s}.$$

Significance

We found the speed directly from the conservation of mechanical energy, without having to solve the differential equation for the motion of a pendulum (see **Oscillations**). We can approach this problem in terms of bar graphs of total energy. Initially, the particle has all potential energy, being at the highest point, and no kinetic energy. When the particle crosses the lowest point at the bottom of the swing, the energy moves from the potential energy column to the kinetic energy column. Therefore, we can imagine a progression of this transfer as the particle moves between its highest point, lowest point of the swing, and back to the highest point (**Figure 8.9**). As the particle travels from the lowest point in the swing to the highest point on the far right hand side of the diagram, the energy bars go in reverse order from (c) to (b) to (a).

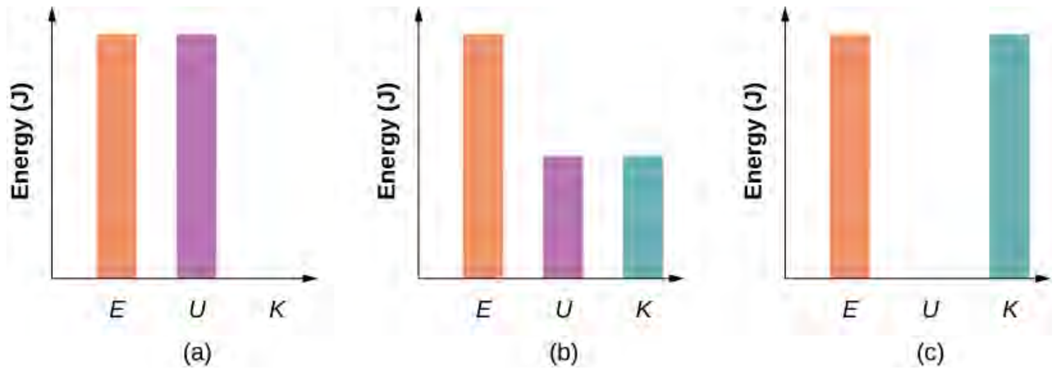


Figure 8.9 Bar graphs representing the total energy (E), potential energy (U), and kinetic energy (K) of the particle in different positions. (a) The total energy of the system equals the potential energy and the kinetic energy is zero, which is found at the highest point the particle reaches. (b) The particle is midway between the highest and lowest point, so the kinetic energy plus potential energy bar graphs equal the total energy. (c) The particle is at the lowest point of the swing, so the kinetic energy bar graph is the highest and equal to the total energy of the system.

8.7 Check Your Understanding How high above the bottom of its arc is the particle in the simple pendulum above, when its speed is 0.81 m/s?

Example 8.8

Air Resistance on a Falling Object

A helicopter is hovering at an altitude of 1 km when a panel from its underside breaks loose and plummets to the ground (**Figure 8.10**). The mass of the panel is 15 kg, and it hits the ground with a speed of 45 m/s. How much mechanical energy was dissipated by air resistance during the panel's descent?

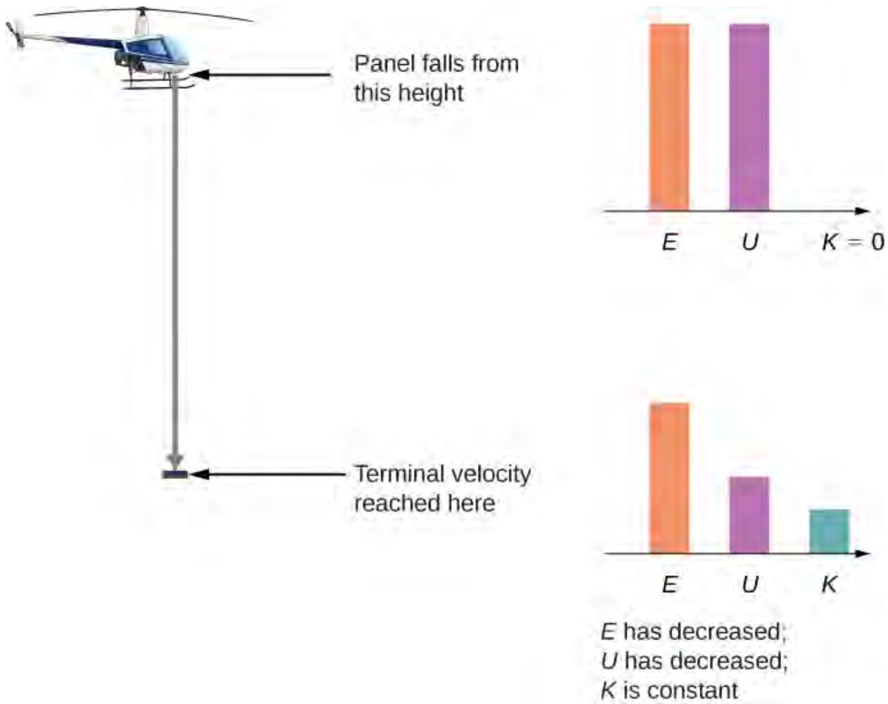


Figure 8.10 A helicopter loses a panel that falls until it reaches terminal velocity of 45 m/s. How much did air resistance contribute to the dissipation of energy in this problem?

Strategy

Step 1: Here only one body is being investigated.

Step 2: Gravitational force is acting on the panel, as well as air resistance, which is stated in the problem.

Step 3: Gravitational force is conservative; however, the non-conservative force of air resistance does negative work on the falling panel, so we can use the conservation of mechanical energy, in the form expressed by **Equation 8.12**, to find the energy dissipated. This energy is the magnitude of the work:

$$\Delta E_{\text{diss}} = \left|W_{\text{nc,if}}\right| = \left|\Delta(K + U)_{\text{if}}\right|.$$

Step 4: The initial kinetic energy, at $y_{\text{i}} = 1 \text{ km}$, is zero. We set the gravitational potential energy to zero at ground level out of convenience.

Step 5: The non-conservative work is set equal to the energies to solve for the work dissipated by air resistance.

Solution

The mechanical energy dissipated by air resistance is the algebraic sum of the gain in the kinetic energy and loss in potential energy. Therefore the calculation of this energy is

$$\begin{aligned} \Delta E_{\text{diss}} &= \left| K_{\text{f}} - K_{\text{i}} + U_{\text{f}} - U_{\text{i}} \right| \\ &= \left| \frac{1}{2}(15\ \text{kg})(45\ \text{m/s})^2 - 0 + 0 - (15\ \text{kg})\left(9.8\ \text{m/s}^2\right)(1000\ \text{m}) \right| = 130\ \text{kJ}. \end{aligned}$$

Significance

Most of the initial mechanical energy of the panel (U_{i}), 147 kJ, was lost to air resistance. Notice that we were able to calculate the energy dissipated without knowing what the force of air resistance was, only that it was dissipative.

8.8 Check Your Understanding You probably recall that, neglecting air resistance, if you throw a projectile straight up, the time it takes to reach its maximum height equals the time it takes to fall from the maximum height back to the starting height. Suppose you cannot neglect air resistance, as in **Example 8.8**. Is the time the projectile takes to go up (a) greater than, (b) less than, or (c) equal to the time it takes to come back down? Explain.

In these examples, we were able to use conservation of energy to calculate the speed of a particle just at particular points in its motion. But the method of analyzing particle motion, starting from energy conservation, is more powerful than that. More advanced treatments of the theory of mechanics allow you to calculate the full time dependence of a particle's motion, for a given potential energy. In fact, it is often the case that a better model for particle motion is provided by the form of its kinetic and potential energies, rather than an equation for force acting on it. (This is especially true for the quantum mechanical description of particles like electrons or atoms.)

We can illustrate some of the simplest features of this energy-based approach by considering a particle in one-dimensional motion, with potential energy $U(x)$ and no non-conservative interactions present. **Equation 8.12** and the definition of velocity require

$$\begin{aligned} K &= \frac{1}{2}mv^2 = E - U(x) \\ v &= \frac{dx}{dt} = \sqrt{\frac{2(E - U(x))}{m}}. \end{aligned}$$

Separate the variables x and t and integrate, from an initial time $t = 0$ to an arbitrary time, to get

$$t = \int_0^t dt = \int_{x_0}^{x} \frac{dt}{\sqrt{2[E - U(x)]/m}}. \tag{8.14}$$

If you can do the integral in **Equation 8.14**, then you can solve for x as a function of t.

Example 8.9

Constant Acceleration

Use the potential energy $U(x) = -E(x/x_0)$, for $E > 0$, in **Equation 8.14** to find the position x of a particle as a function of time t.

Strategy

Since we know how the potential energy changes as a function of x, we can substitute for $U(x)$ in **Equation 8.14**, integrate, and then solve for x. This results in an expression of x as a function of time with constants of energy E, mass m, and the initial position x_0.

Solution

Following the first two suggested steps in the above strategy,

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{(2E/mx_0)(x_0 - x)}} = \frac{1}{\sqrt{(2E/mx_0)}} \left| -2\sqrt{(x_0 - x)} \right|_{x_0}^{x} = -\frac{2\sqrt{(x_0 - x)}}{\sqrt{(2E/mx_0)}}.$$

Solving for the position, we obtain $x(t) = x_0 - \frac{1}{2}(E/mx_0)t^2$.

Significance

The position as a function of time, for this potential, represents one-dimensional motion with constant acceleration, $a = (E/mx_0)$, starting at rest from position x_0. This is not so surprising, since this is a potential energy for a constant force, $F = -dU/dx = E/x_0$, and $a = F/m$.

8.9 Check Your Understanding What potential energy $U(x)$ can you substitute in **Equation 8.13** that will result in motion with constant velocity of 2 m/s for a particle of mass 1 kg and mechanical energy 1 J?

We will look at another more physically appropriate example of the use of **Equation 8.13** after we have explored some further implications that can be drawn from the functional form of a particle's potential energy.

Systems with Several Particles or Objects

Systems generally consist of more than one particle or object. However, the conservation of mechanical energy, in one of the forms in **Equation 8.12** or **Equation 8.13**, is a fundamental law of physics and applies to any system. You just have to include the kinetic and potential energies of all the particles, and the work done by all the non-conservative forces acting on them. Until you learn more about the dynamics of systems composed of many particles, in **Linear Momentum and Collisions**, **Fixed-Axis Rotation**, and **Angular Momentum**, it is better to postpone discussing the application of energy conservation to then.

8.4 | Potential Energy Diagrams and Stability

Learning Objectives

By the end of this section, you will be able to:

- Create and interpret graphs of potential energy
- Explain the connection between stability and potential energy

Often, you can get a good deal of useful information about the dynamical behavior of a mechanical system just by interpreting a graph of its potential energy as a function of position, called a **potential energy diagram**. This is most easily accomplished for a one-dimensional system, whose potential energy can be plotted in one two-dimensional graph—for example, $U(x)$ versus x—on a piece of paper or a computer program. For systems whose motion is in more than one dimension, the motion needs to be studied in three-dimensional space. We will simplify our procedure for one-dimensional motion only.

First, let's look at an object, freely falling vertically, near the surface of Earth, in the absence of air resistance. The mechanical energy of the object is conserved, $E = K + U$, and the potential energy, with respect to zero at ground level, is $U(y) = mgy$, which is a straight line through the origin with slope mg. In the graph shown in **Figure 8.11**, the x-axis is the height above the ground y and the y-axis is the object's energy.

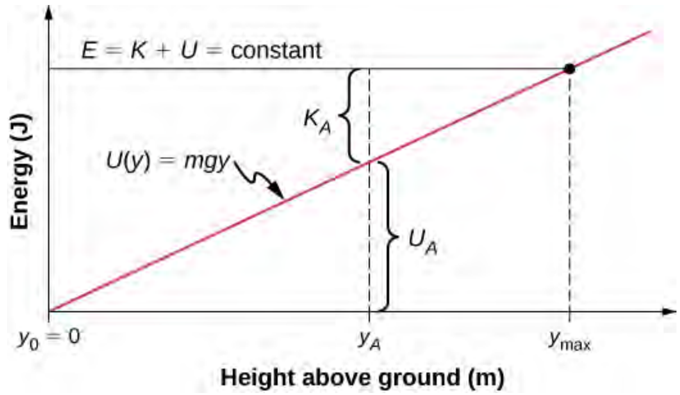


Figure 8.11 The potential energy graph for an object in vertical free fall, with various quantities indicated.

The line at energy E represents the constant mechanical energy of the object, whereas the kinetic and potential energies, K_A and U_A, are indicated at a particular height y_A. You can see how the total energy is divided between kinetic and potential energy as the object's height changes. Since kinetic energy can never be negative, there is a maximum potential energy and a maximum height, which an object with the given total energy cannot exceed:

$$K = E - U \geq 0,$$
$$U \leq E.$$

If we use the gravitational potential energy reference point of zero at y_0, we can rewrite the gravitational potential energy U as mgy. Solving for y results in

$$y \leq E/mg = y_{\text{max}}.$$

We note in this expression that the quantity of the total energy divided by the weight (mg) is located at the maximum height of the particle, or y_{max}. At the maximum height, the kinetic energy and the speed are zero, so if the object were initially traveling upward, its velocity would go through zero there, and y_{max} would be a turning point in the motion. At ground level, $y_0 = 0$, the potential energy is zero, and the kinetic energy and the speed are maximum:

$$\begin{aligned} U_0 &= 0 = E - K_0, \\ E &= K_0 = \frac{1}{2}m{v_0}^2, \\ v_0 &= \pm\sqrt{2E/m}. \end{aligned}$$

The maximum speed $\pm v_0$ gives the initial velocity necessary to reach y_{max}, the maximum height, and $-v_0$ represents the final velocity, after falling from y_{max}. You can read all this information, and more, from the potential energy diagram we have shown.

Consider a mass-spring system on a frictionless, stationary, horizontal surface, so that gravity and the normal contact force do no work and can be ignored (**Figure 8.12**). This is like a one-dimensional system, whose mechanical energy E is a constant and whose potential energy, with respect to zero energy at zero displacement from the spring's unstretched length, $x = 0$, is $U(x) = \frac{1}{2}kx^2$.

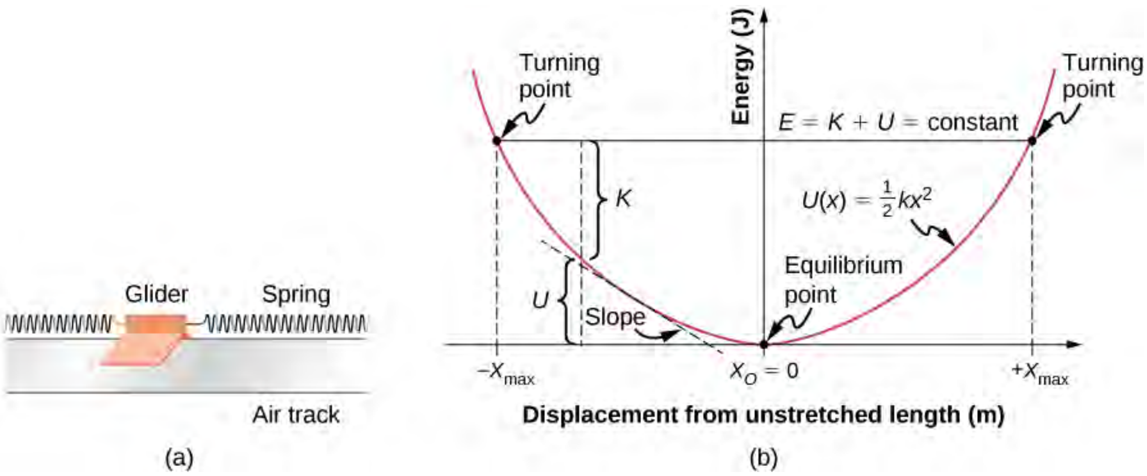


Figure 8.12 (a) A glider between springs on an air track is an example of a horizontal mass-spring system. (b) The potential energy diagram for this system, with various quantities indicated.

You can read off the same type of information from the potential energy diagram in this case, as in the case for the body in vertical free fall, but since the spring potential energy describes a variable force, you can learn more from this graph. As for the object in vertical free fall, you can deduce the physically allowable range of motion and the maximum values of distance and speed, from the limits on the kinetic energy, $0 \le K \le E$. Therefore, $K = 0$ and $U = E$ at a **turning point**, of which there are two for the elastic spring potential energy,

$$x_{\max} = \pm\sqrt{2E/k}.$$

The glider's motion is confined to the region between the turning points, $-x_{\max} \le x \le x_{\max}$. This is true for any (positive) value of E because the potential energy is unbounded with respect to x. For this reason, as well as the shape of the potential energy curve, $U(x)$ is called an infinite potential well. At the bottom of the potential well, $x = 0, U = 0$ and the kinetic energy is a maximum, $K = E$, so $v_{\max} = \pm\sqrt{2E/m}$.

However, from the slope of this potential energy curve, you can also deduce information about the force on the glider and its acceleration. We saw earlier that the negative of the slope of the potential energy is the spring force, which in this case is also the net force, and thus is proportional to the acceleration. When $x = 0$, the slope, the force, and the acceleration are all zero, so this is an **equilibrium point**. The negative of the slope, on either side of the equilibrium point, gives a force pointing back to the equilibrium point, $F = \pm kx$, so the equilibrium is termed stable and the force is called a restoring force. This implies that $U(x)$ has a relative minimum there. If the force on either side of an equilibrium point has a direction opposite from that direction of position change, the equilibrium is termed unstable, and this implies that $U(x)$ has a relative maximum there.

Example 8.10

Quartic and Quadratic Potential Energy Diagram

The potential energy for a particle undergoing one-dimensional motion along the x-axis is $U(x) = 2(x^4 - x^2)$, where U is in joules and x is in meters. The particle is not subject to any non-conservative forces and its mechanical energy is constant at $E = -0.25\,\text{J}$. (a) Is the motion of the particle confined to any regions on the x-axis, and if so, what are they? (b) Are there any equilibrium points, and if so, where are they and are they stable or unstable?

Strategy

First, we need to graph the potential energy as a function of x. The function is zero at the origin, becomes negative as x increases in the positive or negative directions (x^2 is larger than x^4 for $x < 1$), and then becomes positive at sufficiently large $|x|$. Your graph should look like a double potential well, with the zeros determined by solving the equation $U(x) = 0$, and the extremes determined by examining the first and second derivatives of $U(x)$, as shown in **Figure 8.13**.

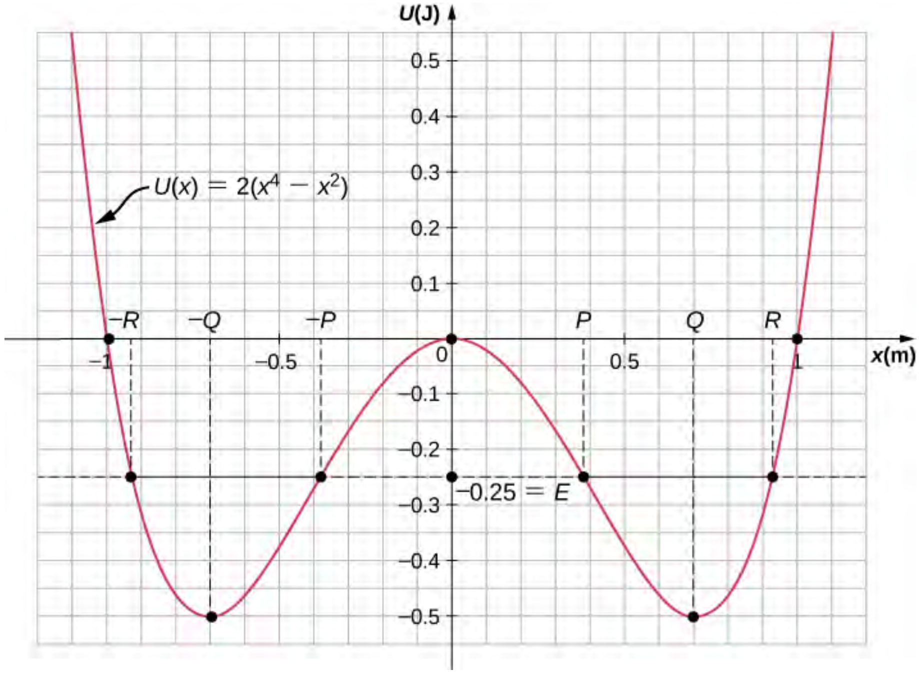


Figure 8.13 The potential energy graph for a one-dimensional, quartic and quadratic potential energy, with various quantities indicated.

You can find the values of (a) the allowed regions along the x-axis, for the given value of the mechanical energy, from the condition that the kinetic energy can't be negative, and (b) the equilibrium points and their stability from the properties of the force (stable for a relative minimum and unstable for a relative maximum of potential energy).

You can just eyeball the graph to reach qualitative answers to the questions in this example. That, after all, is the value of potential energy diagrams. You can see that there are two allowed regions for the motion $(E > U)$ and three equilibrium points (slope $dU/dx = 0$), of which the central one is unstable $\left(d^2U/dx^2 < 0\right)$, and the other two are stable $\left(d^2U/dx^2 > 0\right)$.

Solution

a. To find the allowed regions for x, we use the condition

$$K = E - U = -\frac{1}{4} - 2\left(x^4 - x^2\right) \geq 0.$$

If we complete the square in x^2, this condition simplifies to $2\left(x^2 - \frac{1}{2}\right)^2 \leq \frac{1}{4}$, which we can solve to obtain

$$\frac{1}{2} - \sqrt{\frac{1}{8}} \leq x^2 \leq \frac{1}{2} + \sqrt{\frac{1}{8}}.$$

This represents two allowed regions, $x_p \leq x \leq x_R$ and $-x_R \leq x \leq -x_p$, where $x_p = 0.38$ and $x_R = 0.92$ (in meters).

b. To find the equilibrium points, we solve the equation

$$dU/dx = 8x^3 - 4x = 0$$

and find $x = 0$ and $x = \pm x_Q$, where $x_Q = 1/\sqrt{2} = 0.707$ (meters). The second derivative

$$d^2 U/dx^2 = 24x^2 - 4$$

is negative at $x = 0$, so that position is a relative maximum and the equilibrium there is unstable. The second derivative is positive at $x = \pm x_Q$, so these positions are relative minima and represent stable equilibria.

Significance

The particle in this example can oscillate in the allowed region about either of the two stable equilibrium points we found, but it does not have enough energy to escape from whichever potential well it happens to initially be in. The conservation of mechanical energy and the relations between kinetic energy and speed, and potential energy and force, enable you to deduce much information about the qualitative behavior of the motion of a particle, as well as some quantitative information, from a graph of its potential energy.

8.10 Check Your Understanding Repeat **Example 8.10** when the particle's mechanical energy is $+0.25\text{ J}$.

Before ending this section, let's practice applying the method based on the potential energy of a particle to find its position as a function of time, for the one-dimensional, mass-spring system considered earlier in this section.

Example 8.11

Sinusoidal Oscillations

Find $x(t)$ for a particle moving with a constant mechanical energy $E > 0$ and a potential energy $U(x) = \frac{1}{2}kx^2$, when the particle starts from rest at time $t = 0$.

Strategy

We follow the same steps as we did in **Example 8.9**. Substitute the potential energy U into **Equation 8.14** and factor out the constants, like m or k. Integrate the function and solve the resulting expression for position, which is now a function of time.

Solution

Substitute the potential energy in **Equation 8.14** and integrate using an integral solver found on a web search:

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{(k/m)\left[(2E/k) - x^2\right]}} = \sqrt{\frac{m}{k}}\left[\sin^{-1}\left(\frac{x}{\sqrt{2E/k}}\right) - \sin^{-1}\left(\frac{x_0}{\sqrt{2E/k}}\right)\right].$$

From the initial conditions at $t = 0$, the initial kinetic energy is zero and the initial potential energy is $\frac{1}{2}kx_0{}^2 = E$, from which you can see that $x_0/\sqrt{(2E/k)} = \pm 1$ and $\sin^{-1}(\pm) = \pm 90^0$. Now you can solve for x:

$$x(t) = \sqrt{(2E/k)}\sin\left[\left(\sqrt{k/m}\right)t \pm 90^0\right] = \pm\sqrt{(2E/k)}\cos\left[\left(\sqrt{k/m}\right)t\right].$$

Significance

A few paragraphs earlier, we referred to this mass-spring system as an example of a harmonic oscillator. Here, we anticipate that a harmonic oscillator executes sinusoidal oscillations with a maximum displacement of $\sqrt{(2E/k)}$

(called the amplitude) and a rate of oscillation of $(1/2\pi)\sqrt{k/m}$ (called the frequency). Further discussions about oscillations can be found in **Oscillations**.

8.11 Check Your Understanding Find $x(t)$ for the mass-spring system in **Example 8.11** if the particle starts from $x_0 = 0$ at $t = 0$. What is the particle's initial velocity?

8.5 | Sources of Energy

Learning Objectives

By the end of this section, you will be able to:

- Describe energy transformations and conversions in general terms
- Explain what it means for an energy source to be renewable or nonrenewable

In this chapter, we have studied energy. We learned that energy can take different forms and can be transferred from one form to another. You will find that energy is discussed in many everyday, as well as scientific, contexts, because it is involved in all physical processes. It will also become apparent that many situations are best understood, or most easily conceptualized, by considering energy. So far, no experimental results have contradicted the conservation of energy. In fact, whenever measurements have appeared to conflict with energy conservation, new forms of energy have been discovered or recognized in accordance with this principle.

What are some other forms of energy? Many of these are covered in later chapters (also see **Figure 8.14**), but let's detail a few here:

- Atoms and molecules inside all objects are in random motion. The internal kinetic energy from these random motions is called *thermal energy*, because it is related to the temperature of the object. Note that thermal energy can also be transferred from one place to another, not transformed or converted, by the familiar processes of conduction, convection, and radiation. In this case, the energy is known as *heat energy*.
- *Electrical energy* is a common form that is converted to many other forms and does work in a wide range of practical situations.
- Fuels, such as gasoline and food, have *chemical energy*, which is potential energy arising from their molecular structure. Chemical energy can be converted into thermal energy by reactions like oxidation. Chemical reactions can also produce electrical energy, such as in batteries. Electrical energy can, in turn, produce thermal energy and light, such as in an electric heater or a light bulb.
- Light is just one kind of electromagnetic radiation, or *radiant energy*, which also includes radio, infrared, ultraviolet, X-rays, and gamma rays. All bodies with thermal energy can radiate energy in electromagnetic waves.
- *Nuclear energy* comes from reactions and processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into radiant energy in the Sun, into thermal energy in the boilers of nuclear power plants, and then into electrical energy in the generators of power plants. These and all other forms of energy can be transformed into one another and, to a certain degree, can be converted into mechanical work.

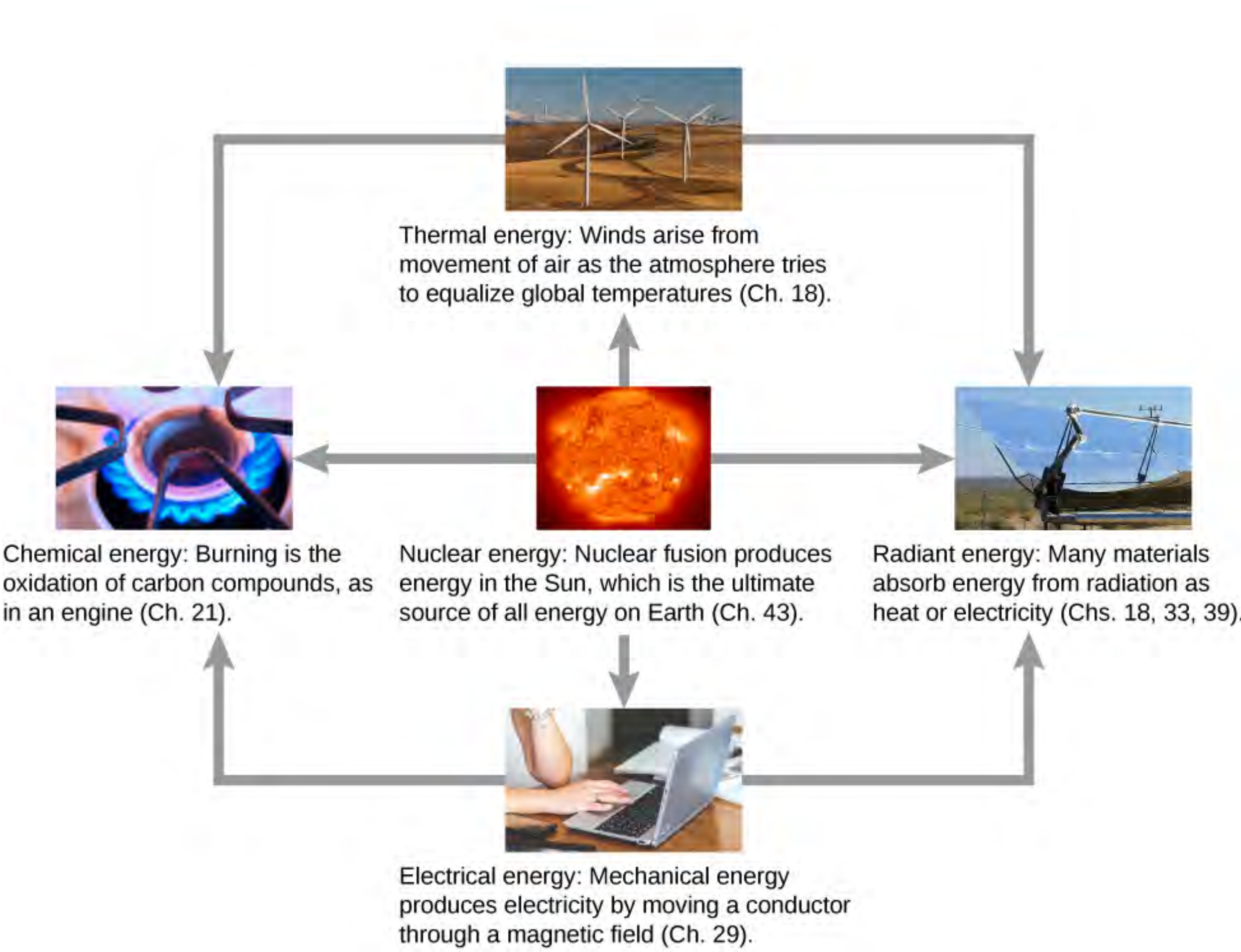


Figure 8.14 Energy that we use in society takes many forms, which be converted from one into another depending on the process involved. We will study many of these forms of energy in later chapters in this text. (credit "sun": EIT SOHO Consortium, ESA, NASA; credit "solar panels": "kjkolb"/Wikimedia Commons; credit "gas burner": Steven Depolo)

The transformation of energy from one form into another happens all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell produces electricity, which can be used to run electric motors or heat water. In an example encompassing many steps, the chemical energy contained in coal is converted into thermal energy as it burns in a furnace, to transform water into steam, in a boiler. Some of the thermal energy in the steam is then converted into mechanical energy as it expands and spins a turbine, which is connected to a generator to produce electrical energy. In these examples, not all of the initial energy is converted into the forms mentioned, because some energy is always transferred to the environment.

Energy is an important element at all levels of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. The principal energy resources used in the world are shown in **Figure 8.15**. The figure distinguishes between two major types of energy sources: **renewable** and **non-renewable**, and further divides each type into a few more specific kinds. Renewable sources are energy sources that are replenished through naturally occurring, ongoing processes, on a time scale that is much shorter than the anticipated lifetime of the civilization using the source. Non-renewable sources are depleted once some of the energy they contain is extracted and converted into other kinds of energy. The natural processes by which non-renewable sources are formed typically take place over geological time scales.

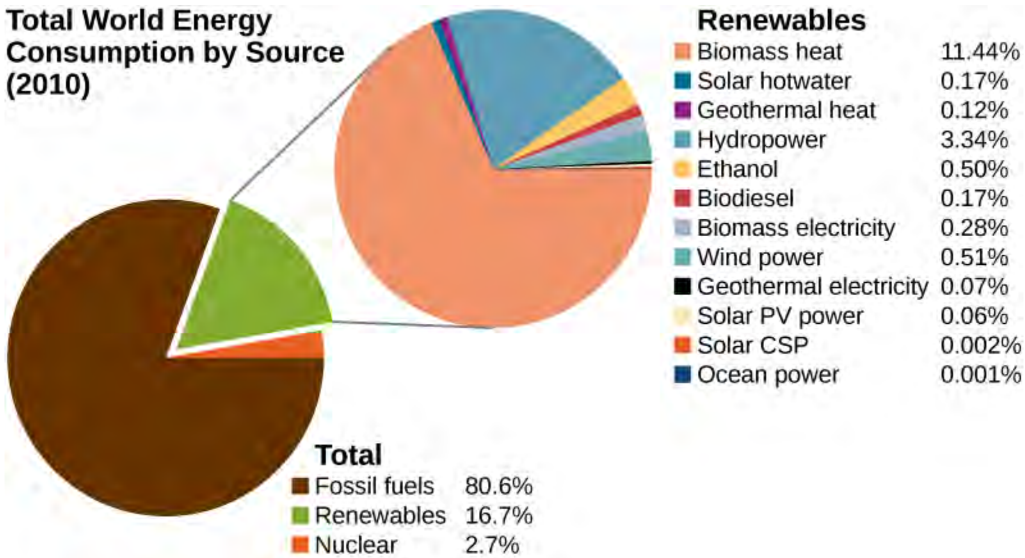


Figure 8.15 World energy consumption by source; the percentage of renewables is increasing, accounting for 19% in 2012.

Our most important non-renewable energy sources are fossil fuels, such as coal, petroleum, and natural gas. These account for about 81% of the world's energy consumption, as shown in the figure. Burning fossil fuels creates chemical reactions that transform potential energy, in the molecular structures of the reactants, into thermal energy and products. This thermal energy can be used to heat buildings or to operate steam-driven machinery. Internal combustion and jet engines convert some of the energy of rapidly expanding gases, released from burning gasoline, into mechanical work. Electrical power generation is mostly derived from transferring energy in expanding steam, via turbines, into mechanical work, which rotates coils of wire in magnetic fields to generate electricity. Nuclear energy is the other non-renewable source shown in **Figure 8.15** and supplies about 3% of the world's consumption. Nuclear reactions release energy by transforming potential energy, in the structure of nuclei, into thermal energy, analogous to energy release in chemical reactions. The thermal energy obtained from nuclear reactions can be transferred and converted into other forms in the same ways that energy from fossil fuels are used.

An unfortunate byproduct of relying on energy produced from the combustion of fossil fuels is the release of carbon dioxide into the atmosphere and its contribution to global warming. Nuclear energy poses environmental problems as well, including the safety and disposal of nuclear waste. Besides these important consequences, reserves of non-renewable sources of energy are limited and, given the rapidly growing rate of world energy consumption, may not last for more than a few hundred years. Considerable effort is going on to develop and expand the use of renewable sources of energy, involving a significant percentage of the world's physicists and engineers.

Four of the renewable energy sources listed in **Figure 8.15**—those using material from plants as fuel (biomass heat, ethanol, biodiesel, and biomass electricity)—involve the same types of energy transformations and conversions as just discussed for fossil and nuclear fuels. The other major types of renewable energy sources are hydropower, wind power, geothermal power, and solar power.

Hydropower is produced by converting the gravitational potential energy of falling or flowing water into kinetic energy and then into work to run electric generators or machinery. Converting the mechanical energy in ocean surface waves and tides is in development. Wind power also converts kinetic energy into work, which can be used directly to generate electricity, operate mills, and propel sailboats.

The interior of Earth has a great deal of thermal energy, part of which is left over from its original formation (gravitational potential energy converted into thermal energy) and part of which is released from radioactive minerals (a form of natural nuclear energy). It will take a very long time for this geothermal energy to escape into space, so people generally regard it as a renewable source, when actually, it's just inexhaustible on human time scales.

The source of solar power is energy carried by the electromagnetic waves radiated by the Sun. Most of this energy is carried by visible light and infrared (heat) radiation. When suitable materials absorb electromagnetic waves, radiant energy is converted into thermal energy, which can be used to heat water, or when concentrated, to make steam and generate electricity (**Figure 8.16**). However, in another important physical process, known as the photoelectric effect, energetic radiation impinging on certain materials is directly converted into electricity. Materials that do this are called photovoltaics (PV in **Figure 8.15**). Some solar power systems use lenses or mirrors to concentrate the Sun's rays, before converting their energy through photovoltaics, and these are qualified as CSP in Figure 8.15.

Figure 8.16 Solar cell arrays found in a sunny area converting the solar energy into stored electrical energy. (credit: Sarah Swenty)

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As we mentioned earlier, the “law of conservation of energy” is a very useful principle in analyzing physical processes. It cannot be proven from basic principles but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system always remains constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through reducing activities (e.g., turning down thermostats, diving fewer kilometers) and/or increasing conversion efficiencies in the performance of a particular task, such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed, created, or generated, you might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat, that is, work that has been “degraded” in the energy transformation. We will discuss this idea in more detail in the chapters on thermodynamics.

CHAPTER 8 REVIEW

KEY TERMS

conservative force force that does work independent of path

conserved quantity one that cannot be created or destroyed, but may be transformed between different forms of itself

energy conservation total energy of an isolated system is constant

equilibrium point position where the assumed conservative, net force on a particle, given by the slope of its potential energy curve, is zero

exact differential is the total differential of a function and requires the use of partial derivatives if the function involves more than one dimension

mechanical energy sum of the kinetic and potential energies

non-conservative force force that does work that depends on path

non-renewable energy source that is not renewable, but is depleted by human consumption

potential energy function of position, energy possessed by an object relative to the system considered

potential energy diagram graph of a particle's potential energy as a function of position

potential energy difference negative of the work done acting between two points in space

renewable energy source that is replenished by natural processes, over human time scales

turning point position where the velocity of a particle, in one-dimensional motion, changes sign

KEY EQUATIONS

Difference of potential energy	$\Delta U_{AB} = U_B - U_A = -W_{AB}$
Potential energy with respect to zero of potential energy at	$\vec{\mathbf{r}}_0 \Delta U = U(\vec{\mathbf{r}}) - U(\vec{\mathbf{r}}_0)$
Gravitational potential energy near Earth's surface	$U(y) = mgy + \text{const.}$
Potential energy for an ideal spring	$U(x) = \frac{1}{2}kx^2 + \text{const.}$
Work done by conservative force over a closed path	$W_{\text{closed path}} = \oint \vec{\mathbf{E}}_{\text{cons}} \cdot d\vec{\mathbf{r}} = 0$
Condition for conservative force in two dimensions	$\left(\frac{dF_x}{dy}\right) = \left(\frac{dF_y}{dx}\right)$
Conservative force is the negative derivative of potential energy	$F_l = -\frac{dU}{dl}$
Conservation of energy with no non-conservative forces	$0 = W_{nc,\,AB} = \Delta(K + U)_{AB} = \Delta E_{AB}.$

SUMMARY

8.1 Potential Energy of a System

- For a single-particle system, the difference of potential energy is the opposite of the work done by the forces acting on the particle as it moves from one position to another.
- Since only differences of potential energy are physically meaningful, the zero of the potential energy function can be chosen at a convenient location.

- The potential energies for Earth's constant gravity, near its surface, and for a Hooke's law force are linear and quadratic functions of position, respectively.

8.2 Conservative and Non-Conservative Forces

- A conservative force is one for which the work done is independent of path. Equivalently, a force is conservative if the work done over any closed path is zero.
- A non-conservative force is one for which the work done depends on the path.
- For a conservative force, the infinitesimal work is an exact differential. This implies conditions on the derivatives of the force's components.
- The component of a conservative force, in a particular direction, equals the negative of the derivative of the potential energy for that force, with respect to a displacement in that direction.

8.3 Conservation of Energy

- A conserved quantity is a physical property that stays constant regardless of the path taken.
- A form of the work-energy theorem says that the change in the mechanical energy of a particle equals the work done on it by non-conservative forces.
- If non-conservative forces do no work and there are no external forces, the mechanical energy of a particle stays constant. This is a statement of the conservation of mechanical energy and there is no change in the total mechanical energy.
- For one-dimensional particle motion, in which the mechanical energy is constant and the potential energy is known, the particle's position, as a function of time, can be found by evaluating an integral that is derived from the conservation of mechanical energy.

8.4 Potential Energy Diagrams and Stability

- Interpreting a one-dimensional potential energy diagram allows you to obtain qualitative, and some quantitative, information about the motion of a particle.
- At a turning point, the potential energy equals the mechanical energy and the kinetic energy is zero, indicating that the direction of the velocity reverses there.
- The negative of the slope of the potential energy curve, for a particle, equals the one-dimensional component of the conservative force on the particle. At an equilibrium point, the slope is zero and is a stable (unstable) equilibrium for a potential energy minimum (maximum).

8.5 Sources of Energy

- Energy can be transferred from one system to another and transformed or converted from one type into another. Some of the basic types of energy are kinetic, potential, thermal, and electromagnetic.
- Renewable energy sources are those that are replenished by ongoing natural processes, over human time scales. Examples are wind, water, geothermal, and solar power.
- Non-renewable energy sources are those that are depleted by consumption, over human time scales. Examples are fossil fuel and nuclear power.

CONCEPTUAL QUESTIONS

8.1 Potential Energy of a System

1. The kinetic energy of a system must always be positive or zero. Explain whether this is true for the potential energy of a system.

2. The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer drives from it, starting just before the swimmer steps on the board until just after his feet leave it.

3. Describe the gravitational potential energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

4. A couple of soccer balls of equal mass are kicked off the ground at the same speed but at different angles. Soccer ball A is kicked off at an angle slightly above the horizontal, whereas ball B is kicked slightly below the vertical. How do each of the following compare for ball A and ball B? (a) The initial kinetic energy and (b) the change in gravitational potential energy from the ground to the highest point? If the energy in part (a) differs from part (b), explain why there is a difference between the two energies.

5. What is the dominant factor that affects the speed of an object that started from rest down a frictionless incline if the only work done on the object is from gravitational forces?

6. Two people observe a leaf falling from a tree. One person is standing on a ladder and the other is on the ground. If each person were to compare the energy of the leaf observed, would each person find the following to be the same or different for the leaf, from the point where it falls off the tree to when it hits the ground: (a) the kinetic energy of the leaf; (b) the change in gravitational potential energy; (c) the final gravitational potential energy?

8.2 Conservative and Non-Conservative Forces

7. What is the physical meaning of a non-conservative force?

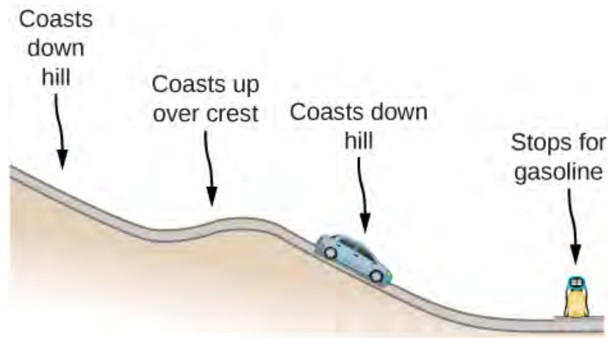
8. A bottle rocket is shot straight up in the air with a speed 30 m/s. If the air resistance is ignored, the bottle would go up to a height of approximately 46 m. However, the rocket goes up to only 35 m before returning to the ground. What happened? Explain, giving only a qualitative response.

9. An external force acts on a particle during a trip from one point to another and back to that same point. This particle is only effected by conservative forces. Does this particle's kinetic energy and potential energy change as a result of this trip?

8.3 Conservation of Energy

10. When a body slides down an inclined plane, does the work of friction depend on the body's initial speed? Answer the same question for a body sliding down a curved surface.

11. Consider the following scenario. A car for which friction is *not* negligible accelerates from rest down a hill, running out of gasoline after a short distance (see below). The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events.

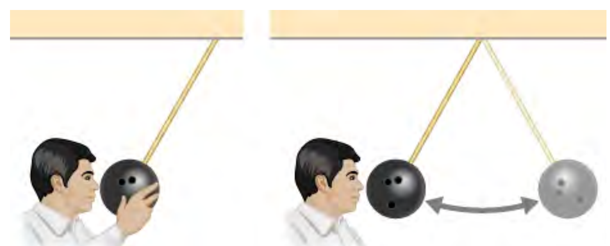


12. A dropped ball bounces to one-half its original height. Discuss the energy transformations that take place.

13. "$E = K + U$ constant is a special case of the work-energy theorem." Discuss this statement.

14. In a common physics demonstration, a bowling ball is suspended from the ceiling by a rope.

The professor pulls the ball away from its equilibrium position and holds it adjacent to his nose, as shown below. He releases the ball so that it swings directly away from him. Does he get struck by the ball on its return swing? What is he trying to show in this demonstration?



15. A child jumps up and down on a bed, reaching a higher height after each bounce. Explain how the child can increase his maximum gravitational potential energy with each bounce.

16. Can a non-conservative force increase the mechanical energy of the system?

17. Neglecting air resistance, how much would I have to raise the vertical height if I wanted to double the impact speed of a falling object?

18. A box is dropped onto a spring at its equilibrium position. The spring compresses with the box attached and comes to rest. Since the spring is in the vertical position, does the change in the gravitational potential energy of the box while the spring is compressing need to be considered in this problem?

PROBLEMS

8.1 Potential Energy of a System

19. Using values from **Table 8.2**, how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous X-rays. Later-model tube TVs had shielding that absorbed X-rays before they escaped and exposed viewers.)

20. If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from **Table 8.1**)?

21. A camera weighing 10 N falls from a small drone hovering $20\,\text{m}$ overhead and enters free fall. What is the gravitational potential energy change of the camera from the drone to the ground if you take a reference point of (a) the ground being zero gravitational potential energy? (b) The drone being zero gravitational potential energy? What is the gravitational potential energy of the camera (c) before it falls from the drone and (d) after the camera lands on the ground if the reference point of zero gravitational potential energy is taken to be a second person looking out of a building $30\,\text{m}$ from the ground?

22. Someone drops a $50 - \text{g}$ pebble off of a docked cruise ship, $70.0\,\text{m}$ from the water line. A person on a dock $3.0\,\text{m}$ from the water line holds out a net to catch the pebble. (a) How much work is done on the pebble by gravity during the drop? (b) What is the change in the gravitational potential energy during the drop? If the gravitational potential energy is zero at the water line, what is the gravitational potential energy (c) when the pebble is dropped? (d) When it reaches the net? What if the gravitational potential energy was 30.0 Joules at water level? (e) Find the answers to the same questions in (c) and (d).

23. A cat's crinkle ball toy of mass $15\,\text{g}$ is thrown straight up with an initial speed of $3\,\text{m/s}$. Assume in this problem that air drag is negligible. (a) What is the kinetic energy of the ball as it leaves the hand? (b) How much work is done by the gravitational force during the ball's rise to its peak? (c) What is the change in the gravitational potential energy of the ball during the rise to its peak? (d) If the gravitational potential energy is taken to be zero at the point where it leaves your hand, what is the gravitational potential energy when it reaches the maximum height? (e) What if the gravitational potential energy is taken to be zero at the maximum height the ball reaches, what would the gravitational potential energy be when it leaves the hand? (f) What is the maximum height the ball reaches?

8.2 Conservative and Non-Conservative Forces

24. A force $F(x) = (3.0/x)\,\text{N}$ acts on a particle as it moves along the positive x-axis. (a) How much work does the force do on the particle as it moves from $x = 2.0\,\text{m}$ to $x = 5.0\,\text{m}$? (b) Picking a convenient reference point of the potential energy to be zero at $x = \infty$, find the potential energy for this force.

25. A force $F(x) = \left(-5.0x^2 + 7.0x\right)\text{N}$ acts on a particle. (a) How much work does the force do on the particle as it moves from $x = 2.0\,\text{m}$ to $x = 5.0\,\text{m}$? (b) Picking a convenient reference point of the potential energy to be zero at $x = \infty$, find the potential energy for this force.

26. Find the force corresponding to the potential energy $U(x) = -a/x + b/x^2$.

27. The potential energy function for either one of the two atoms in a diatomic molecule is often approximated by $U(x) = -a/x^{12} - b/x^6$ where x is the distance between the atoms. (a) At what distance of seperation does the potential energy have a local minimum (not at $x = \infty$)? (b) What is the force on an atom at this separation? (c) How does the force vary with the separation distance?

28. A particle of mass $2.0\,\text{kg}$ moves under the influence of the force $F(x) = (3/\sqrt{x})\,\text{N}$. If its speed at $x = 2.0\,\text{m}$ is $v = 6.0\,\text{m/s}$, what is its speed at $x = 7.0\,\text{m}$?

29. A particle of mass $2.0\,\text{kg}$ moves under the influence of the force $F(x) = \left(-5x^2 + 7x\right)\text{N}$. If its speed at $x = -4.0\,\text{m}$ is $v = 20.0\,\text{m/s}$, what is its speed at $x = 4.0\,\text{m}$?

30. A crate on rollers is being pushed without frictional loss of energy across the floor of a freight car (see the following figure). The car is moving to the right with a constant speed v_0. If the crate starts at rest relative to the freight car, then from the work-energy theorem, $Fd = mv^2/2$, where d, the distance the crate moves, and v, the speed of the crate, are both measured relative to the freight car. (a) To an observer at rest beside the tracks, what distance d' is the crate pushed when it moves the distance d in the car? (b) What are the crate's initial and final speeds

v_0' and v' as measured by the observer beside the tracks? (c) Show that $Fd' = m(v')^2/2 - m(v'_0)^2/2$ and, consequently, that work is equal to the change in kinetic energy in both reference systems.

8.3 Conservation of Energy

31. A boy throws a ball of mass $0.25\,\text{kg}$ straight upward with an initial speed of $20\,\text{m/s}$ When the ball returns to the boy, its speed is $17\,\text{m/s}$ How much much work does air resistance do on the ball during its flight?

32. A mouse of mass 200 g falls 100 m down a vertical mine shaft and lands at the bottom with a speed of 8.0 m/s. During its fall, how much work is done on the mouse by air resistance?

33. Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown. (*Hint:* show that $K_\text{i} + U_\text{i} = K_\text{f} + U_\text{f}$)

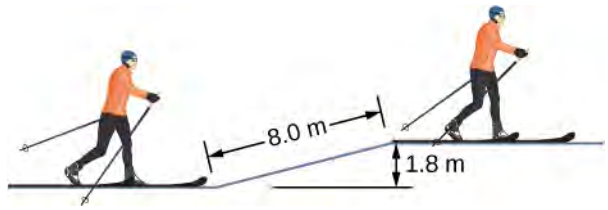
34. A 1.0-kg ball at the end of a 2.0-m string swings in a vertical plane. At its lowest point the ball is moving with a speed of 10 m/s. (a) What is its speed at the top of its path? (b) What is the tension in the string when the ball is at the bottom and at the top of its path?

35. Ignoring details associated with friction, extra forces exerted by arm and leg muscles, and other factors, we can consider a pole vault as the conversion of an athlete's running kinetic energy to gravitational potential energy. If an athlete is to lift his body 4.8 m during a vault, what speed must he have when he plants his pole?

36. Tarzan grabs a vine hanging vertically from a tall tree when he is running at $9.0\,\text{m/s}$. (a) How high can he swing upward? (b) Does the length of the vine affect this height?

37. Assume that the force of a bow on an arrow behaves like the spring force. In aiming the arrow, an archer pulls the bow back 50 cm and holds it in position with a force of $150\,\text{N}$. If the mass of the arrow is $50\,\text{g}$ and the "spring" is massless, what is the speed of the arrow immediately after it leaves the bow?

38. A $100-\text{kg}$ man is skiing across level ground at a speed of $8.0\,\text{m/s}$ when he comes to the small slope 1.8 m higher than ground level shown in the following figure. (a) If the skier coasts up the hill, what is his speed when he reaches the top plateau? Assume friction between the snow and skis is negligible. (b) What is his speed when he reaches the upper level if an $80-\text{N}$ frictional force acts on the skis?

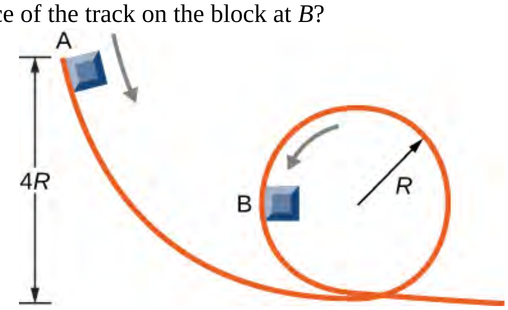


39. A sled of mass 70 kg starts from rest and slides down a $10°$ incline $80\,\text{m}$ long. It then travels for 20 m horizontally before starting back up an $8°$ incline. It travels 80 m along this incline before coming to rest. What is the net work done on the sled by friction?

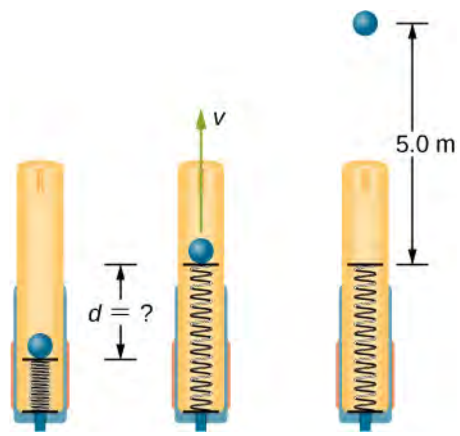
40. A girl on a skateboard (total mass of 40 kg) is moving at a speed of 10 m/s at the bottom of a long ramp. The ramp is inclined at $20°$ with respect to the horizontal. If she travels 14.2 mupward along the ramp before stopping, what is the net frictional force on her?

41. A baseball of mass 0.25 kg is hit at home plate with a speed of 40 m/s. When it lands in a seat in the left-field bleachers a horizontal distance 120 m from home plate, it is moving at 30 m/s. If the ball lands 20 m above the spot where it was hit, how much work is done on it by air resistance?

42. A small block of mass *m* slides without friction around the loop-the-loop apparatus shown below. (a) If the block starts from rest at *A*, what is its speed at *B*? (b) What is the force of the track on the block at *B*?



43. The massless spring of a spring gun has a force constant $k = 12\,\text{N/cm}$. When the gun is aimed vertically, a 15-g projectile is shot to a height of 5.0 m above the end of the expanded spring. (See below.) How much was the spring compressed initially?



44. A small ball is tied to a string and set rotating with negligible friction in a vertical circle. Prove that the tension in the string at the bottom of the circle exceeds that at the top of the circle by eight times the weight of the ball. Assume the ball's speed is zero as it sails over the top of the circle and there is no additional energy added to the ball during rotation.

8.4 Potential Energy Diagrams and Stability

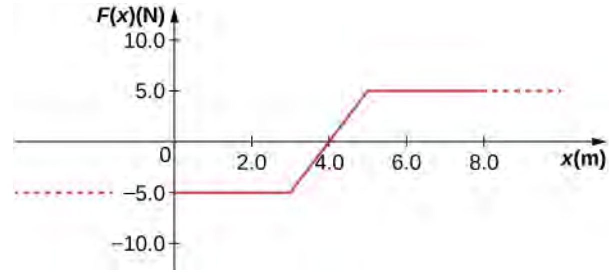
45. A mysterious constant force of 10 N acts horizontally on everything. The direction of the force is found to be always pointed toward a wall in a big hall. Find the potential energy of a particle due to this force when it is at a distance *x* from the wall, assuming the potential energy at the wall to be zero.

46. A single force $F(x) = -4.0x$ (in newtons) acts on a 1.0-kg body. When $x = 3.5\text{ m},$ the speed of the body is 4.0 m/s. What is its speed at $x = 2.0\text{ m}$?

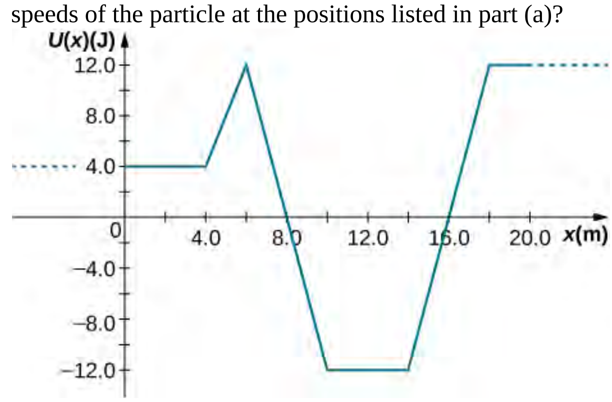
47. A particle of mass 4.0 kg is constrained to move along the *x*-axis under a single force $F(x) = -cx^3,$ where $c = 8.0\text{ N/m}^3.$ The particle's speed at *A*, where $x_A = 1.0\text{ m},$ is 6.0 m/s. What is its speed at *B*, where $x_B = -2.0\text{ m}$?

48. The force on a particle of mass 2.0 kg varies with position according to $F(x) = -3.0x^2$ (*x* in meters, *F*(*x*) in newtons). The particle's velocity at $x = 2.0\text{ m}$ is 5.0 m/s. Calculate the mechanical energy of the particle using (a) the origin as the reference point and (b) $x = 4.0\text{ m}$ as the reference point. (c) Find the particle's velocity at $x = 1.0\text{ m}.$ Do this part of the problem for each reference point.

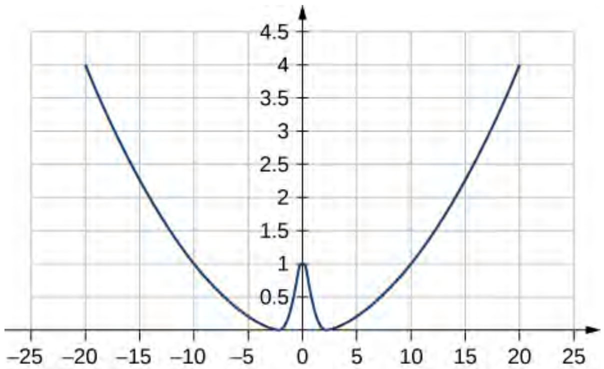
49. A 4.0-kg particle moving along the *x*-axis is acted upon by the force whose functional form appears below. The velocity of the particle at $x = 0$ is $v = 6.0\text{ m/s}.$ Find the particle's speed at $x =$ (a)$2.0\text{ m},$ (b)$4.0\text{ m},$ (c)$10.0\text{ m},$ (d) Does the particle turn around at some point and head back toward the origin? (e) Repeat part (d) if $v = 2.0\text{ m/s}$ at $x = 0.$



50. A particle of mass 0.50 kg moves along the *x*-axis with a potential energy whose dependence on *x* is shown below. (a) What is the force on the particle at $x = 2.0,$ 5.0, 8.0, and 12 m? (b) If the total mechanical energy *E* of the particle is −6.0 J, what are the minimum and maximum positions of the particle? (c) What are these positions if $E = 2.0\text{ J}$? (d) If $E = 16\text{ J}$, what are the speeds of the particle at the positions listed in part (a)?



51. (a) Sketch a graph of the potential energy function $U(x) = kx^2/2 + Ae^{-\alpha x^2},$ where $k,$ $A,$ and α are constants. (b) What is the force corresponding to this potential energy? (c) Suppose a particle of mass *m* moving with this potential energy has a velocity v_a when its position is $x = a$. Show that the particle does not pass through the origin unless $A \leq \dfrac{m v_a{}^2 + ka^2}{2\left(1 - e^{-\alpha a^2}\right)}.$



8.5 Sources of Energy

52. In the cartoon movie ***Pocahontas* (https://openstaxcollege.org/l/21pocahontclip)**, Pocahontas runs to the edge of a cliff and jumps off, showcasing the fun side of her personality. (a) If she is running at 3.0 m/s before jumping off the cliff and she hits the water at the bottom of the cliff at 20.0 m/s, how high is the cliff? Assume negligible air drag in this cartoon. (b) If she jumped off the same cliff from a standstill, how fast would she be falling right before she hit the water?

53. In the reality television show **"Amazing Race" (https://openstaxcollege.org/l/21amazraceclip)**, a contestant is firing 12-kg watermelons from a slingshot to hit targets down the field. The slingshot is pulled back 1.5 m and the watermelon is considered to be at ground level. The launch point is 0.3 m from the ground and the targets are 10 m horizontally away. Calculate the spring constant of the slingshot.

54. In the **Back to the Future movies (https://openstaxcollege.org/l/21bactofutclip)**, a DeLorean car of mass 1230 kg travels at 88 miles per hour to venture back to the future. (a) What is the kinetic energy of the DeLorian? (b) What spring constant would be needed to stop this DeLorean in a distance of 0.1m?

55. In the **Hunger Games movie (https://openstaxcollege.org/l/21HungGamesclip)**, Katniss Everdeen fires a 0.0200-kg arrow from ground level to pierce an apple up on a stage. The spring constant of the bow is 330 N/m and she pulls the arrow back a distance of 0.55 m. The apple on the stage is 5.00 m higher than the launching point of the arrow. At what speed does the arrow (a) leave the bow? (b) strike the apple?

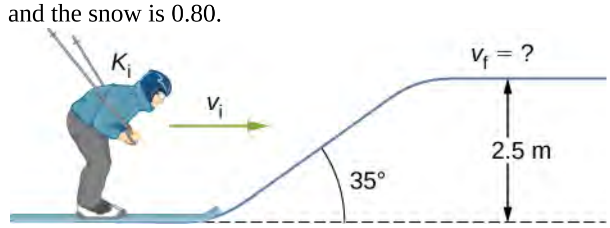
56. In a **"Top Fail" video (https://openstaxcollege.org/l/21topfailvideo)**, two women run at each other and collide by hitting exercise balls together. If each woman has a mass of 50 kg, which includes the exercise ball, and one woman runs to the right at 2.0 m/s and the other is running toward her at 1.0 m/s, (a) how much total kinetic energy is there in the system? (b) If energy is conserved after the collision and each exercise ball has a mass of 2.0 kg, how fast would the balls fly off toward the camera?

57. In a **Coyote/Road Runner cartoon clip (https://openstaxcollege.org/l/21coyroadcarcl)**, a spring expands quickly and sends the coyote into a rock. If the spring extended 5 m and sent the coyote of mass 20 kg to a speed of 15 m/s, (a) what is the spring constant of this spring? (b) If the coyote were sent vertically into the air with the energy given to him by the spring, how high could he go if there were no non-conservative forces?

58. In an iconic movie scene, **Forrest Gump (https://openstaxcollege.org/l/21ForrGumpvid)** runs around the country. If he is running at a constant speed of 3 m/s, would it take him more or less energy to run uphill or downhill and why?

59. In the movie ***Monty Python and the Holy Grail* (https://openstaxcollege.org/l/21monpytmovcl)** a cow is catapulted from the top of a castle wall over to the people down below. The gravitational potential energy is set to zero at ground level. The cow is launched from a spring of spring constant 1.1×10^4 N/m that is expanded 0.5 m from equilibrium. If the castle is 9.1 m tall and the mass of the cow is 110 kg, (a) what is the gravitational potential energy of the cow at the top of the castle? (b) What is the elastic spring energy of the cow before the catapult is released? (c) What is the speed of the cow right before it lands on the ground?

60. A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m high rise as shown. Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.80.

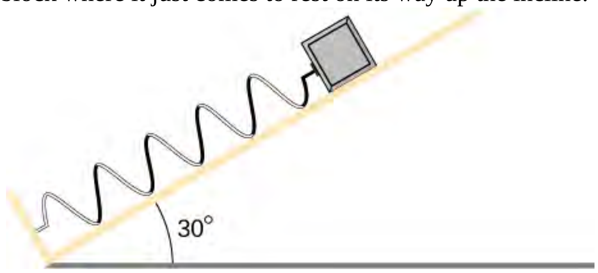


61. (a) How high a hill can a car coast up (engines disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope of $2.5°$ above the horizontal?

62. A 5.00×10^5 -kg subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the spring constant k of the spring?

63. A pogo stick has a spring with a spring constant of 2.5×10^4 N/m, which can be compressed 12.0 cm. To what maximum height from the uncompressed spring can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40 kg?

64. A block of mass 500 g is attached to a spring of spring constant 80 N/m (see the following figure). The other end of the spring is attached to a support while the mass rests on a rough surface with a coefficient of friction of 0.20 that is inclined at angle of $30°$. The block is pushed along the surface till the spring compresses by 10 cm and is then released from rest. (a) How much potential energy was stored in the block-spring-support system when the block was just released? (b) Determine the speed of the block when it crosses the point when the spring is neither compressed nor stretched. (c) Determine the position of the block where it just comes to rest on its way up the incline.

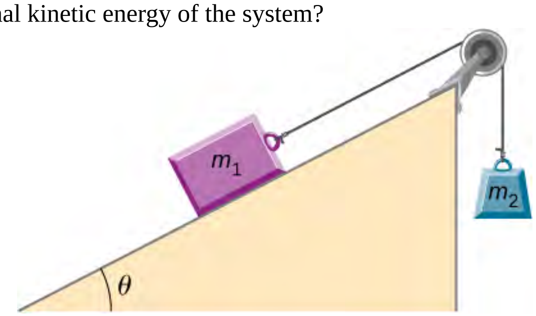


65. A block of mass 200 g is attached at the end of a massless spring of spring constant 100 N/cm. The other end of the spring is attached to the ceiling and the mass is brought to rest. Let us mark this point as *O*. Suppose, this point is taken to be the zero of the potential energy of the block, both from the weight and the spring force. The mass hangs freely and the spring is in a stretched state. The block is then pulled downward by another 5.00 cm and released from rest. (a) What is the net potential energy of the block at the instant the block is at the lowest point? (b) What is the net potential energy of the block at the instant the block returns to the point marked *O*? (c) What is the speed of the block as it crosses the point marked *O*? (d) How high above the point marked *O* does the block rise before coming to rest again?

66. A T-shirt cannon launches a shirt at 5.00 m/s from a platform height of 3.00 m from ground level. How fast will the shirt be traveling if it is caught by someone whose hands are (a) 1.00 m from ground level? (b) 4.00 m from ground level? Neglect air drag.

67. A child (32 kg) jumps up and down on a trampoline. The trampoline exerts a spring restoring force on the child with a constant of 5000 N/m. At the highest point of the bounce, the child is 1.0 m above the level surface of the trampoline. What is the compression distance of the trampoline? Neglect the bending of the legs or any transfer of energy of the child into the trampoline while jumping.

68. Shown below is a box of mass m_1 that sits on a frictionless incline at an angle above the horizontal θ. This box is connected by a relatively massless string, over a frictionless pulley, and finally connected to a box at rest over the ledge, labeled m_2. If m_1 and m_2 are a height *h* above the ground and $m_2 >> m_1$: (a) What is the initial gravitational potential energy of the system? (b) What is the final kinetic energy of the system?

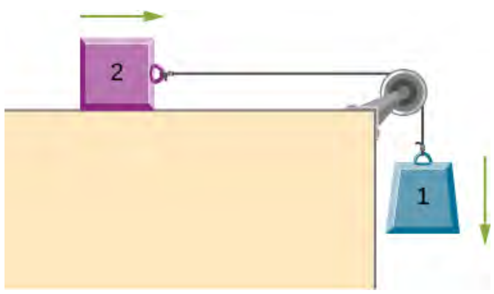


ADDITIONAL PROBLEMS

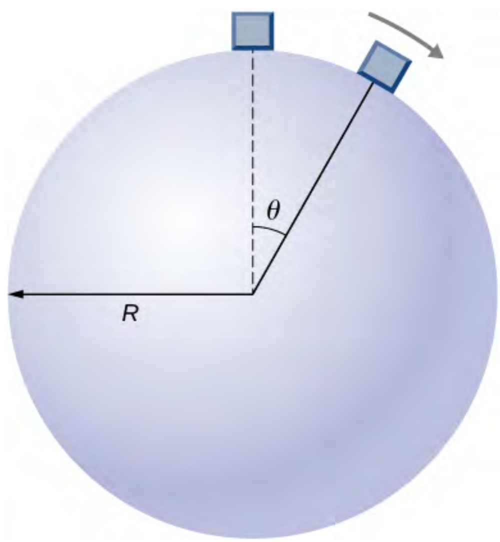
69. A massless spring with force constant $k = 200\,\text{N/m}$ hangs from the ceiling. A 2.0-kg block is attached to the free end of the spring and released. If the block falls 17 cm before starting back upwards, how much work is done by friction during its descent?

70. A particle of mass 2.0 kg moves under the influence of the force $F(x) = \left(-5x^2 + 7x\right)\text{N}$. Suppose a frictional force also acts on the particle. If the particle's speed when it starts at $x = -4.0\,\text{m}$ is 0.0 m/s and when it arrives at $x = 4.0\,\text{m}$ is 9.0 m/s, how much work is done on it by the frictional force between $x = -4.0\,\text{m}$ and $x = 4.0\,\text{m}$?

71. Block 2 shown below slides along a frictionless table as block 1 falls. Both blocks are attached by a frictionless pulley. Find the speed of the blocks after they have each moved 2.0 m. Assume that they start at rest and that the pulley has negligible mass. Use $m_1 = 2.0\,\text{kg}$ and $m_2 = 4.0\,\text{kg}$.

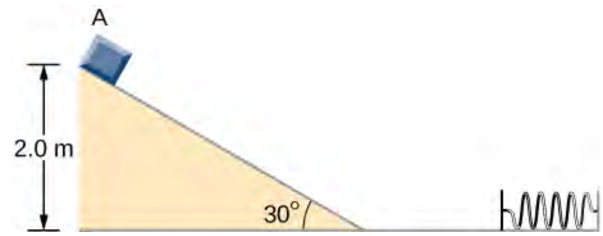


72. A body of mass *m* and negligible size starts from rest and slides down the surface of a frictionless solid sphere of radius *R*. (See below.) Prove that the body leaves the sphere when $\theta = \cos^{-1}(2/3)$.

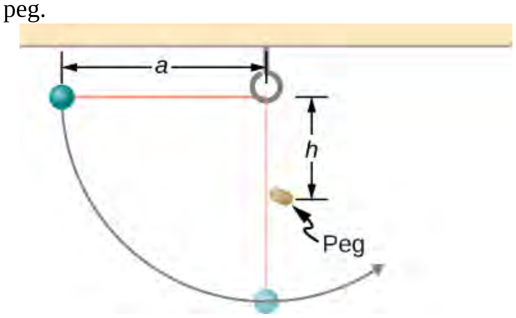


73. A mysterious force acts on all particles along a particular line and always points towards a particular point *P* on the line. The magnitude of the force on a particle increases as the cube of the distance from that point; that is $F \infty r^3$, if the distance from *P* to the position of the particle is *r*. Let *b* be the proportionality constant, and write the magnitude of the force as $F = br^3$. Find the potential energy of a particle subjected to this force when the particle is at a distance *D* from *P*, assuming the potential energy to be zero when the particle is at *P*.

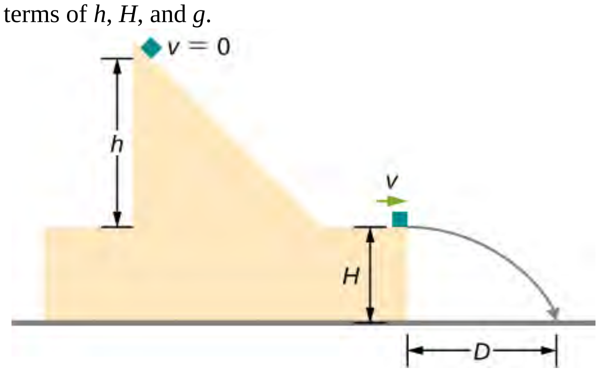
74. An object of mass 10 kg is released at point *A*, slides to the bottom of the $30°$ incline, then collides with a horizontal massless spring, compressing it a maximum distance of 0.75 m. (See below.) The spring constant is 500 M/m, the height of the incline is 2.0 m, and the horizontal surface is frictionless. (a) What is the speed of the object at the bottom of the incline? (b) What is the work of friction on the object while it is on the incline? (c) The spring recoils and sends the object back toward the incline. What is the speed of the object when it reaches the base of the incline? (d) What vertical distance does it move back up the incline?



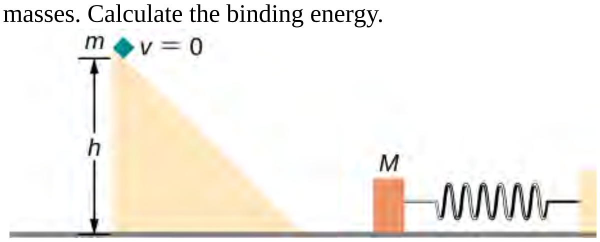
75. Shown below is a small ball of mass *m* attached to a string of length *a*. A small peg is located a distance *h* below the point where the string is supported. If the ball is released when the string is horizontal, show that *h* must be greater than 3*a*/5 if the ball is to swing completely around the peg.



76. A block leaves a frictionless inclined surface horizontally after dropping off by a height *h*. Find the horizontal distance *D* where it will land on the floor, in terms of *h*, *H*, and *g*.



77. A block of mass *m*, after sliding down a frictionless incline, strikes another block of mass *M* that is attached to a spring of spring constant *k* (see below). The blocks stick together upon impact and travel together. (a) Find the compression of the spring in terms of *m*, *M*, *h*, *g*, and *k* when the combination comes to rest. (b) The loss of kinetic energy as a result of the bonding of the two masses upon impact is stored in the so-called binding energy of the two masses. Calculate the binding energy.



78. A block of mass 300 g is attached to a spring of spring constant 100 N/m. The other end of the spring is attached to a support while the block rests on a smooth horizontal table and can slide freely without any friction. The block is pushed horizontally till the spring compresses by 12 cm, and then the block is released from rest. (a) How much potential energy was stored in the block-spring support system when the block was just released? (b) Determine the speed of the block when it crosses the point when the spring is neither compressed nor stretched. (c) Determine the speed of the block when it has traveled a distance of 20 cm from where it was released.

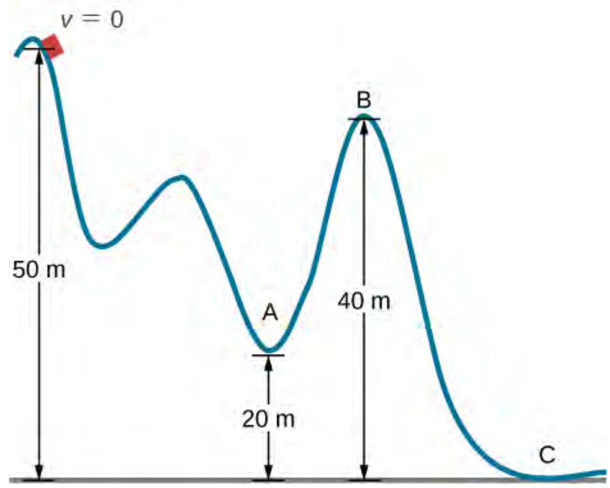
79. Consider a block of mass 0.200 kg attached to a spring of spring constant 100 N/m. The block is placed on a frictionless table, and the other end of the spring is attached to the wall so that the spring is level with the table. The block is then pushed in so that the spring is compressed by 10.0 cm. Find the speed of the block as it crosses (a) the point when the spring is not stretched, (b) 5.00 cm to the left of point in (a), and (c) 5.00 cm to the right of point in (a).

80. A skier starts from rest and slides downhill. What will be the speed of the skier if he drops by 20 meters in vertical height? Ignore any air resistance (which will, in reality, be quite a lot), and any friction between the skis and the snow.

81. Repeat the preceding problem, but this time, suppose that the work done by air resistance cannot be ignored. Let the work done by the air resistance when the skier goes from *A* to *B* along the given hilly path be −2000 J. The work done by air resistance is negative since the air resistance acts in the opposite direction to the displacement. Supposing the mass of the skier is 50 kg, what is the speed of the skier at point *B*?

82. Two bodies are interacting by a conservative force. Show that the mechanical energy of an isolated system consisting of two bodies interacting with a conservative force is conserved. (*Hint*: Start by using Newton's third law and the definition of work to find the work done on each body by the conservative force.)

83. In an amusement park, a car rolls in a track as shown below. Find the speed of the car at *A*, *B*, and *C*. Note that the work done by the rolling friction is zero since the displacement of the point at which the rolling friction acts on the tires is momentarily at rest and therefore has a zero displacement.



84. A 200-g steel ball is tied to a 2.00-m "massless" string and hung from the ceiling to make a pendulum, and then, the ball is brought to a position making a $30°$ angle with the vertical direction and released from rest. Ignoring the effects of the air resistance, find the speed of the ball when the string (a) is vertically down, (b) makes an angle of $20°$ with the vertical and (c) makes an angle of $10°$ with the vertical.

85. A hockey puck is shot across an ice-covered pond. Before the hockey puck was hit, the puck was at rest. After the hit, the puck has a speed of 40 m/s. The puck comes to rest after going a distance of 30 m. (a) Describe how the energy of the puck changes over time, giving the numerical values of any work or energy involved. (b) Find the magnitude of the net friction force.

86. A projectile of mass 2 kg is fired with a speed of 20 m/s at an angle of $30°$ with respect to the horizontal. (a) Calculate the initial total energy of the projectile given that the reference point of zero gravitational potential energy at the launch position. (b) Calculate the kinetic energy at the highest vertical position of the projectile. (c) Calculate the gravitational potential energy at the highest vertical position. (d) Calculate the maximum height that the projectile reaches. Compare this result by solving the same problem using your knowledge of projectile motion.

87. An artillery shell is fired at a target 200 m above the ground. When the shell is 100 m in the air, it has a speed of 100 m/s. What is its speed when it hits its target? Neglect air friction.

88. How much energy is lost to a dissipative drag force if a 60-kg person falls at a constant speed for 15 meters?

89. A box slides on a frictionless surface with a total energy of 50 J. It hits a spring and compresses the spring a distance of 25 cm from equilibrium. If the same box with the same initial energy slides on a rough surface, it only compresses the spring a distance of 15 cm, how much energy must have been lost by sliding on the rough surface?

9 | LINEAR MOMENTUM AND COLLISIONS

Figure 9.1 The concepts of impulse, momentum, and center of mass are crucial for a major-league baseball player to successfully get a hit. If he misjudges these quantities, he might break his bat instead. (credit: modification of work by “Cathy T”/Flickr)

Chapter Outline

Introduction

The concepts of work, energy, and the work-energy theorem are valuable for two primary reasons: First, they are powerful computational tools, making it much easier to analyze complex physical systems than is possible using Newton’s laws directly (for example, systems with nonconstant forces); and second, the observation that the total energy of a closed system is conserved means that the system can only evolve in ways that are consistent with energy conservation. In other words, a system cannot evolve randomly; it can only change in ways that conserve energy.

In this chapter, we develop and define another conserved quantity, called *linear momentum*, and another relationship (the *impulse-momentum theorem*), which will put an additional constraint on how a system evolves in time. Conservation of momentum is useful for understanding collisions, such as that shown in the above image. It is just as powerful, just as important, and just as useful as conservation of energy and the work-energy theorem.

9.1 | Linear Momentum

Learning Objectives

By the end of this section, you will be able to:

- Explain what momentum is, physically
- Calculate the momentum of a moving object

Our study of kinetic energy showed that a complete understanding of an object's motion must include both its mass and its velocity ($K = (1/2)mv^2$). However, as powerful as this concept is, it does not include any information about the direction of the moving object's velocity vector. We'll now define a physical quantity that includes direction.

Like kinetic energy, this quantity includes both mass and velocity; like kinetic energy, it is a way of characterizing the "quantity of motion" of an object. It is given the name **momentum** (from the Latin word *movimentum*, meaning "movement"), and it is represented by the symbol *p*.

Momentum

The momentum *p* of an object is the product of its mass and its velocity:

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}. \tag{9.1}$$

Figure 9.2 The velocity and momentum vectors for the ball are in the same direction. The mass of the ball is about 0.5 kg, so the momentum vector is about half the length of the velocity vector because momentum is velocity time mass. (credit: modification of work by Ben Sutherland)

As shown in **Figure 9.2**, momentum is a vector quantity (since velocity is). This is one of the things that makes momentum useful and not a duplication of kinetic energy. It is perhaps most useful when determining whether an object's motion is difficult to change (**Figure 9.3**) or easy to change (**Figure 9.4**).

Figure 9.3 This supertanker transports a huge mass of oil; as a consequence, it takes a long time for a force to change its (comparatively small) velocity. (credit: modification of work by "the_tahoe_guy"/Flickr)

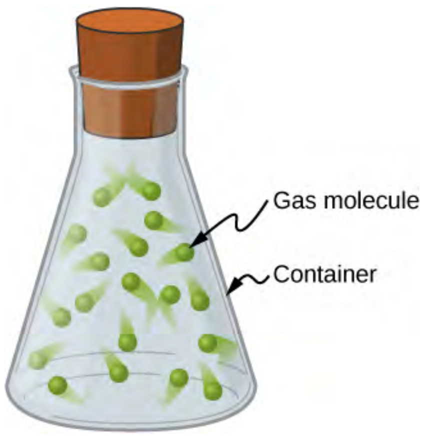


Figure 9.4 Gas molecules can have very large velocities, but these velocities change nearly instantaneously when they collide with the container walls or with each other. This is primarily because their masses are so tiny.

Unlike kinetic energy, momentum depends equally on an object's mass and velocity. For example, as you will learn when you study thermodynamics, the average speed of an air molecule at room temperature is approximately 500 m/s, with an average molecular mass of 6×10^{-25} kg ; its momentum is thus

$$p_{\text{molecule}} = \left(6 \times 10^{-25}\ \text{kg}\right)\left(500\ \frac{\text{m}}{\text{s}}\right) = 3 \times 10^{-22}\ \frac{\text{kg} \cdot \text{m}}{\text{s}}.$$

For comparison, a typical automobile might have a speed of only 15 m/s, but a mass of 1400 kg, giving it a momentum of

$$p_{\text{car}} = (1400\ \text{kg})\left(15\ \frac{\text{m}}{\text{s}}\right) = 21{,}000\ \frac{\text{kg} \cdot \text{m}}{\text{s}}.$$

These momenta are different by 27 orders of magnitude, or a factor of a billion billion billion!

9.2 | Impulse and Collisions

Learning Objectives

By the end of this section, you will be able to:

- Explain what an impulse is, physically
- Describe what an impulse does
- Relate impulses to collisions
- Apply the impulse-momentum theorem to solve problems

We have defined momentum to be the product of mass and velocity. Therefore, if an object's velocity should change (due to the application of a force on the object), then necessarily, its momentum changes as well. This indicates a connection between momentum and force. The purpose of this section is to explore and describe that connection.

Suppose you apply a force on a free object for some amount of time. Clearly, the larger the force, the larger the object's change of momentum will be. Alternatively, the more time you spend applying this force, again the larger the change of momentum will be, as depicted in **Figure 9.5**. The amount by which the object's motion changes is therefore proportional to the magnitude of the force, and also to the time interval over which the force is applied.

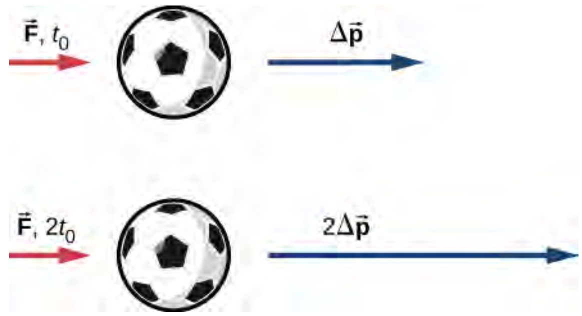


Figure 9.5 The change in momentum of an object is proportional to the length of time during which the force is applied. If a force is exerted on the lower ball for twice as long as on the upper ball, then the change in the momentum of the lower ball is twice that of the upper ball.

Mathematically, if a quantity is proportional to two (or more) things, then it is proportional to the product of those things. The product of a force and a time interval (over which that force acts) is called **impulse**, and is given the symbol $\vec{\mathbf{J}}$.

Impulse

Let $\vec{\mathbf{F}}(t)$ be the force applied to an object over some differential time interval *dt* (**Figure 9.6**). The resulting impulse on the object is defined as

$$d\vec{\mathbf{J}} \equiv \vec{\mathbf{F}}(t)dt. \tag{9.2}$$

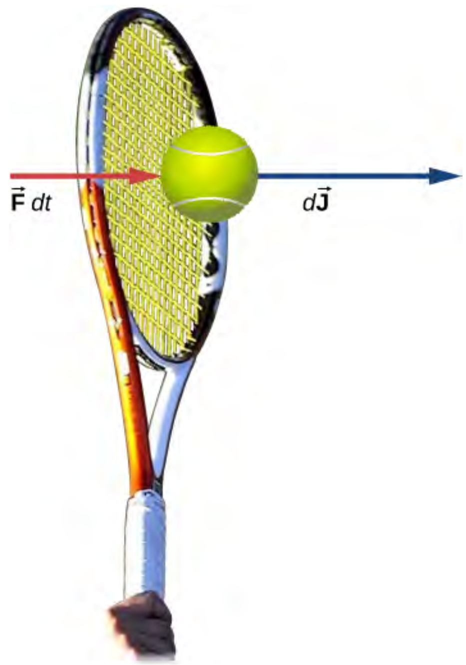


Figure 9.6 A force applied by a tennis racquet to a tennis ball over a time interval generates an impulse acting on the ball.

The total impulse over the interval $t_f - t_i$ is

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} d\vec{\mathbf{J}} \quad \text{or} \quad \vec{\mathbf{J}} \equiv \int_{t_i}^{t_f} \vec{\mathbf{F}}(t)dt. \tag{9.3}$$

Equation 9.2 and **Equation 9.3** together say that when a force is applied for an infinitesimal time interval *dt*, it causes an infinitesimal impulse $d\vec{\mathbf{J}}$, and the total impulse given to the object is defined to be the sum (integral) of all these infinitesimal impulses.

To calculate the impulse using **Equation 9.3**, we need to know the force function *F*(*t*), which we often don't. However, a result from calculus is useful here: Recall that the average value of a function over some interval is calculated by

$$f(x)_{\text{ave}} = \frac{1}{\Delta x}\int_{x_i}^{x_f} f(x)dx$$

where $\Delta x = x_f - x_i$. Applying this to the time-dependent force function, we obtain

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{\mathbf{F}}(t)dt. \tag{9.4}$$

Therefore, from **Equation 9.3**,

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}_{\text{ave}}\Delta t. \tag{9.5}$$

The idea here is that you can calculate the impulse on the object even if you don't know the details of the force as a function of time; you only need the average force. In fact, though, the process is usually reversed: You determine the impulse (by measurement or calculation) and then calculate the average force that caused that impulse.

To calculate the impulse, a useful result follows from writing the force in **Equation 9.3** as $\vec{\mathbf{F}}(t) = m\vec{\mathbf{a}}(t)$:

$$\vec{\mathbf{J}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}(t)dt = m\int_{t_i}^{t_f} \vec{\mathbf{a}}(t)dt = m\left[\vec{\mathbf{v}}(t_f) - \vec{\mathbf{v}}_i\right].$$

For a constant force $\vec{\mathbf{F}}_{ave} = \vec{\mathbf{F}} = m\vec{\mathbf{a}}$, this simplifies to

$$\vec{\mathbf{J}} = m\vec{\mathbf{a}}\Delta t = m\vec{\mathbf{v}}_f - m\vec{\mathbf{v}}_i = m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i).$$

That is,

$$\vec{\mathbf{J}} = m\Delta\vec{\mathbf{v}}. \quad (9.6)$$

Note that the integral form, **Equation 9.3**, applies to constant forces as well; in that case, since the force is independent of time, it comes out of the integral, which can then be trivially evaluated.

Example 9.1

The Arizona Meteor Crater

Approximately 50,000 years ago, a large (radius of 25 m) iron-nickel meteorite collided with Earth at an estimated speed of 1.28×10^4 m/s in what is now the northern Arizona desert, in the United States. The impact produced a crater that is still visible today (**Figure 9.7**); it is approximately 1200 m (three-quarters of a mile) in diameter, 170 m deep, and has a rim that rises 45 m above the surrounding desert plain. Iron-nickel meteorites typically have a density of $\rho = 7970\ \text{kg/m}^3$. Use impulse considerations to estimate the average force and the maximum force that the meteor applied to Earth during the impact.

Figure 9.7 The Arizona Meteor Crater in Flagstaff, Arizona (often referred to as the Barringer Crater after the person who first suggested its origin and whose family owns the land). (credit: “Shane.torgerson”/Wikimedia Commons)

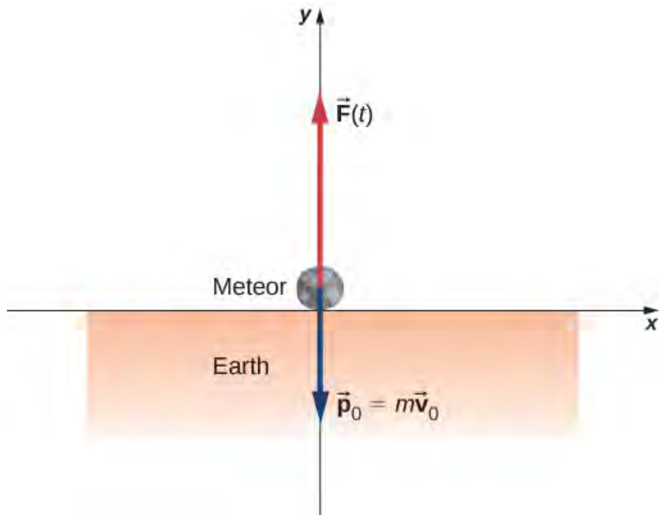
Strategy

It is conceptually easier to reverse the question and calculate the force that Earth applied on the meteor in order to stop it. Therefore, we’ll calculate the force on the meteor and then use Newton’s third law to argue that the force from the meteor on Earth was equal in magnitude and opposite in direction.

Using the given data about the meteor, and making reasonable guesses about the shape of the meteor and impact time, we first calculate the impulse using **Equation 9.6**. We then use the relationship between force and impulse **Equation 9.5** to estimate the average force during impact. Next, we choose a reasonable force function for the impact event, calculate the average value of that function **Equation 9.4**, and set the resulting expression equal to the calculated average force. This enables us to solve for the maximum force.

Solution

Define upward to be the +*y*-direction. For simplicity, assume the meteor is traveling vertically downward prior to impact. In that case, its initial velocity is $\vec{\mathbf{v}}_i = -v_i \hat{\mathbf{j}}$, and the force Earth exerts on the meteor points upward, $\vec{\mathbf{F}}(t) = +F(t)\hat{\mathbf{j}}$. The situation at $t = 0$ is depicted below.



The average force during the impact is related to the impulse by

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\vec{\mathbf{J}}}{\Delta t}.$$

From **Equation 9.6**, $\vec{\mathbf{J}} = m\Delta\vec{\mathbf{v}}$, so we have

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{m\Delta\vec{\mathbf{v}}}{\Delta t}.$$

The mass is equal to the product of the meteor's density and its volume:

$$m = \rho V.$$

If we assume (guess) that the meteor was roughly spherical, we have

$$V = \frac{4}{3}\pi R^3.$$

Thus we obtain

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\rho V\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\rho\left(\frac{4}{3}\pi R^3\right)\left(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i\right)}{\Delta t}.$$

The problem says the velocity at impact was $-1.28 \times 10^4 \text{ m/s}\,\hat{\mathbf{j}}$ (the final velocity is zero); also, we guess that the primary impact lasted about $t_{\max} = 2\text{ s}$. Substituting these values gives

$$\begin{aligned} \vec{\mathbf{F}}_{\text{ave}} &= \frac{\left(7970\,\frac{\text{kg}}{\text{m}^3}\right)\left[\frac{4}{3}\,\pi\,(25\text{ m})^3\right]\left[0\,\frac{\text{m}}{\text{s}} - \left(-1.28\times 10^4\,\frac{\text{m}}{\text{s}}\,\hat{\mathbf{j}}\right)\right]}{2\text{ s}}. \\ &= +\left(3.33\times 10^{12}\text{ N}\right)\hat{\mathbf{j}} \end{aligned}$$

This is the average force applied during the collision. Notice that this force vector points in the same direction as the change of velocity vector $\Delta\vec{\mathbf{v}}$.

Next, we calculate the maximum force. The impulse is related to the force function by

$$\vec{\mathbf{J}} = \int_{t_i}^{t_{\max}} \vec{\mathbf{F}}(t)dt.$$

We need to make a reasonable choice for the force as a function of time. We define $t = 0$ to be the moment the meteor first touches the ground. Then we assume the force is a maximum at impact, and rapidly drops to zero. A function that does this is

$$F(t) = F_{\max}\,e^{-t^2/\left(2\tau^2\right)}.$$

(The parameter τ represents how rapidly the force decreases to zero.) The average force is

$$F_{\text{ave}} = \frac{1}{\Delta t}\int_0^{t_{\max}} F_{\max}e^{-t^2/\left(2\tau^2\right)}dt$$

where $\Delta t = t_{\max} - 0\text{ s}$. Since we already have a numeric value for F_{ave} , we can use the result of the integral to obtain $F_{\max}$.

Choosing $\tau = \frac{1}{e}t_{\max}$ (this is a common choice, as you will see in later chapters), and guessing that $t_{\max} = 2\text{ s}$, this integral evaluates to

$$F_{\text{avg}} = 0.458\,F_{\max}.$$

Thus, the maximum force has a magnitude of

$$\begin{aligned} 0.458F_{\max} &= 3.33\times 10^{12}\text{ N} \\ F_{\max} &= 7.27\times 10^{12}\text{ N} \end{aligned}.$$

The complete force function, including the direction, is

$$\vec{\mathbf{F}}(t) = \left(7.27\times 10^{12}\text{ N}\right)e^{-t^2/\left(8\text{s}^2\right)}\,\hat{\mathbf{y}}.$$

This is the force Earth applied to the meteor; by Newton's third law, the force the meteor applied to Earth is

$$\vec{\mathbf{F}}(t) = -\left(7.27\times 10^{12}\text{ N}\right)e^{-t^2/\left(8\text{s}^2\right)}\,\hat{\mathbf{y}}$$

which is the answer to the original question.

Significance

The graph of this function contains important information. Let's graph (the magnitude of) both this function and the average force together (**Figure 9.8**).

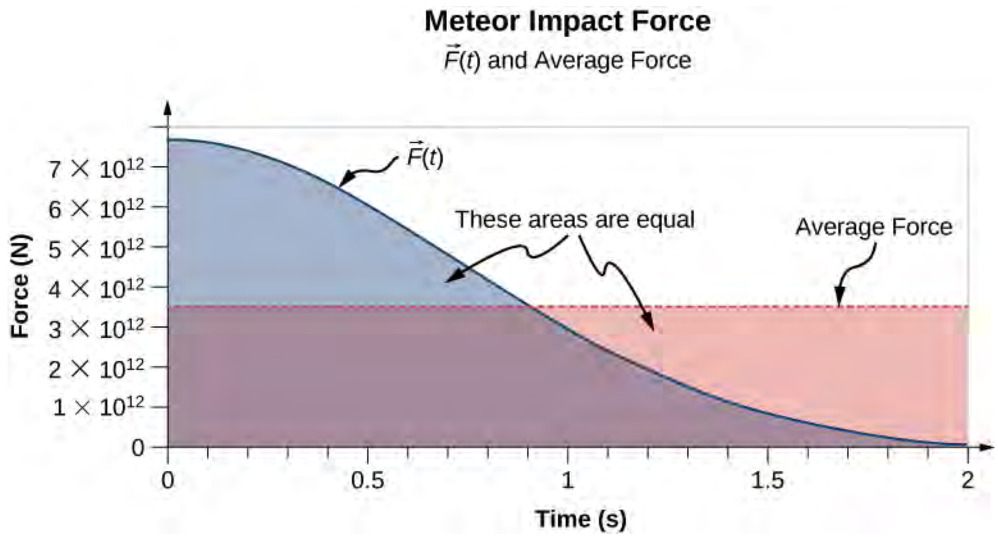


Figure 9.8 A graph of the average force (in red) and the force as a function of time (blue) of the meteor impact. The areas under the curves are equal to each other, and are numerically equal to the applied impulse.

Notice that the area under each plot has been filled in. For the plot of the (constant) force F_{ave}, the area is a rectangle, corresponding to $F_{\text{ave}}\,\Delta t = J$. As for the plot of $F(t)$, recall from calculus that the area under the plot of a function is numerically equal to the integral of that function, over the specified interval; so here, that is $\int_0^{t_{\max}} F(t)dt = J$. Thus, the areas are equal, and both represent the impulse that the meteor applied to Earth during the two-second impact. The average force on Earth sounds like a huge force, and it is. Nevertheless, Earth barely noticed it. The acceleration Earth obtained was just

$$\vec{\mathbf{a}} = \frac{-\vec{\mathbf{F}}_{\text{ave}}}{M_{\text{Earth}}} = \frac{-(3.33\times10^{12}\ \text{N})\hat{\mathbf{j}}}{5.97\times10^{24}\ \text{kg}} = -\left(5.6\times10^{-13}\ \frac{\text{m}}{\text{s}^2}\right)\hat{\mathbf{j}}$$

which is completely immeasurable. That said, the impact created seismic waves that nowadays could be detected by modern monitoring equipment.

Example 9.2

The Benefits of Impulse

A car traveling at 27 m/s collides with a building. The collision with the building causes the car to come to a stop in approximately 1 second. The driver, who weighs 860 N, is protected by a combination of a variable-tension seatbelt and an airbag (**Figure 9.9**). (In effect, the driver collides with the seatbelt and airbag and *not* with the building.) The airbag and seatbelt slow his velocity, such that he comes to a stop in approximately 2.5 s.

a. What average force does the driver experience during the collision?

b. Without the seatbelt and airbag, his collision time (with the steering wheel) would have been approximately 0.20 s. What force would he experience in this case?

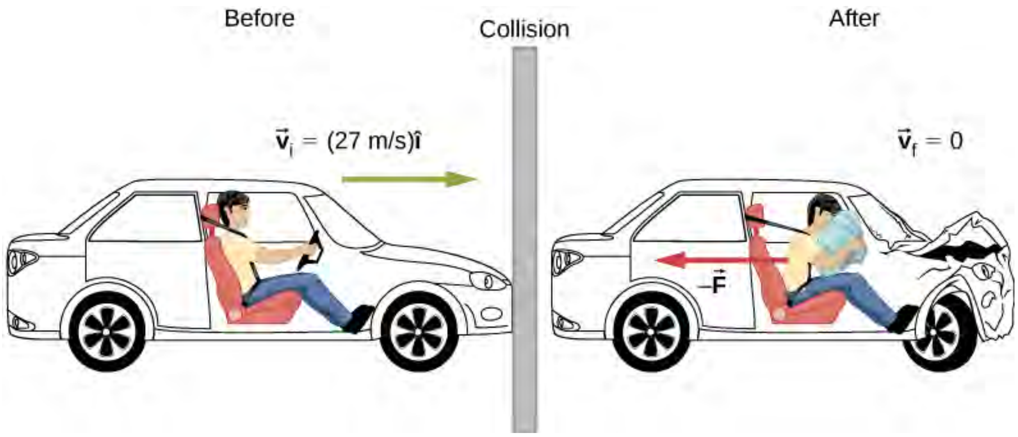


Figure 9.9 The motion of a car and its driver at the instant before and the instant after colliding with the wall. The restrained driver experiences a large backward force from the seatbelt and airbag, which causes his velocity to decrease to zero. (The forward force from the seatback is much smaller than the backward force, so we neglect it in the solution.)

Strategy

We are given the driver's weight, his initial and final velocities, and the time of collision; we are asked to calculate a force. Impulse seems the right way to tackle this; we can combine **Equation 9.5** and **Equation 9.6**.

Solution

a. Define the +*x*-direction to be the direction the car is initially moving. We know

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}\,\Delta t$$

and

$$\vec{\mathbf{J}} = m\Delta\vec{\mathbf{v}}.$$

Since J is equal to both those things, they must be equal to each other:

$$\vec{\mathbf{F}}\,\Delta t = m\Delta\vec{\mathbf{v}}.$$

We need to convert this weight to the equivalent mass, expressed in SI units:

$$\frac{860\text{ N}}{9.8\text{ m/s}^2} = 87.8\text{ kg}.$$

Remembering that $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i$, and noting that the final velocity is zero, we solve for the force:

$$\vec{\mathbf{F}} = m\frac{0 - v_i\,\hat{\mathbf{i}}}{\Delta t} = (87.8\text{ kg})\left(\frac{-(27\text{ m/s})\,\hat{\mathbf{i}}}{2.5\text{ s}}\right) = -(948\text{ N})\,\hat{\mathbf{i}}.$$

The negative sign implies that the force slows him down. For perspective, this is about 1.1 times his own weight.

b. Same calculation, just the different time interval:

$$\vec{\mathbf{F}} = (87.8\text{ kg})\left(\frac{-(27\text{ m/s})\,\hat{\mathbf{i}}}{0.20\text{ s}}\right) = -(11{,}853\text{ N})\,\hat{\mathbf{i}}$$

which is about 14 times his own weight. Big difference!

Significance

You see that the value of an airbag is how greatly it reduces the force on the vehicle occupants. For this reason, they have been required on all passenger vehicles in the United States since 1991, and have been commonplace throughout Europe and Asia since the mid-1990s. The change of momentum in a crash is the same, with or without an airbag; the force, however, is vastly different.

Effect of Impulse

Since an impulse is a force acting for some amount of time, it causes an object's motion to change. Recall **Equation 9.6**:

$$\vec{\mathbf{J}} = m\Delta\vec{\mathbf{v}}.$$

Because $m\vec{\mathbf{v}}$ is the momentum of a system, $m\Delta\vec{\mathbf{v}}$ is the *change* of momentum $\Delta\vec{\mathbf{p}}$. This gives us the following relation, called the **impulse-momentum theorem** (or relation).

Impulse-Momentum Theorem

An impulse applied to a system changes the system's momentum, and that change of momentum is exactly equal to the impulse that was applied:

$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}. \quad (9.7)$$

The impulse-momentum theorem is depicted graphically in **Figure 9.10**.

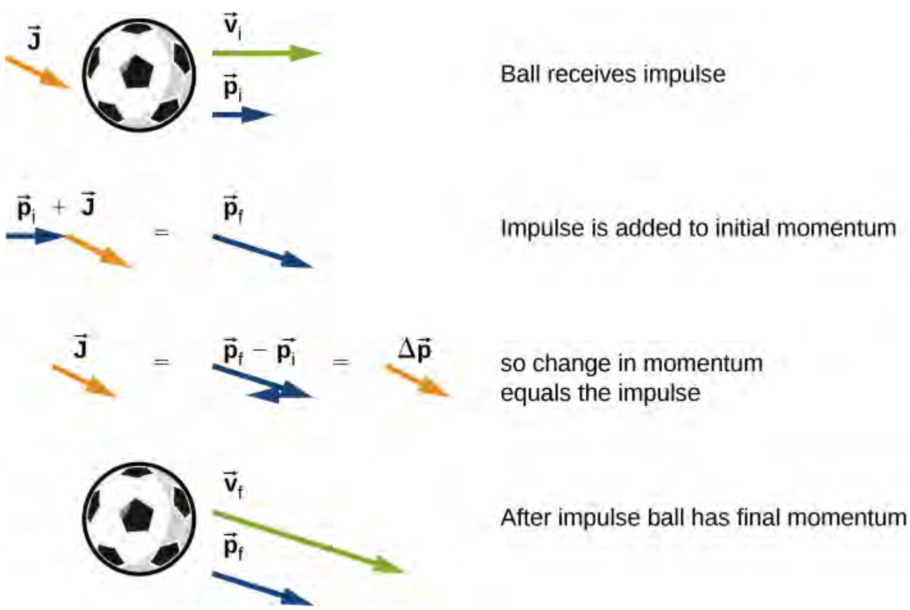


Figure 9.10 Illustration of impulse-momentum theorem. (a) A ball with initial velocity $\vec{\mathbf{v}}_0$ and momentum $\vec{\mathbf{p}}_0$ receives an impulse $\vec{\mathbf{J}}$. (b) This impulse is added vectorially to the initial momentum. (c) Thus, the impulse equals the change in momentum, $\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}$. (d) After the impulse, the ball moves off with its new momentum $\vec{\mathbf{p}}_f$.

There are two crucial concepts in the impulse-momentum theorem:

1. Impulse is a vector quantity; an impulse of, say, $-(10\,\text{N}\cdot\text{s})\,\hat{\mathbf{i}}$ is very different from an impulse of $+(10\,\text{N}\cdot\text{s})\,\hat{\mathbf{i}}$; they cause completely opposite changes of momentum.

2. An impulse does not cause momentum; rather, it causes a *change* in the momentum of an object. Thus, you must subtract the final momentum from the initial momentum, and—since momentum is also a vector quantity—you must take careful account of the signs of the momentum vectors.

The most common questions asked in relation to impulse are to calculate the applied force, or the change of velocity that occurs as a result of applying an impulse. The general approach is the same.

Problem-Solving Strategy: Impulse-Momentum Theorem

1. Express the impulse as force times the relevant time interval.
2. Express the impulse as the change of momentum, usually $m\Delta v$.
3. Equate these and solve for the desired quantity.

Example 9.3

Moving the *Enterprise*

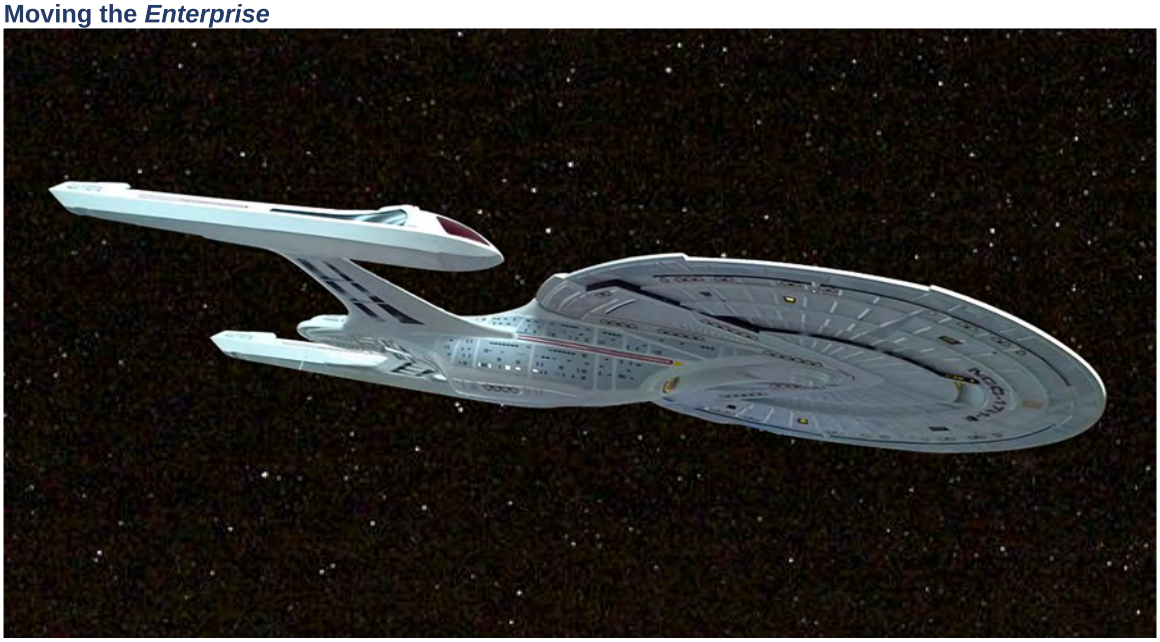


Figure 9.11 The fictional starship *Enterprise* from the Star Trek adventures operated on so-called "impulse engines" that combined matter with antimatter to produce energy.

"Mister Sulu, take us out; ahead one-quarter impulse." With this command, Captain Kirk of the starship *Enterprise* (**Figure 9.11**) has his ship start from rest to a final speed of $v_{\text{f}} = 1/4\left(3.0 \times 10^8 \text{ m/s}\right)$. Assuming this maneuver is completed in 60 s, what average force did the impulse engines apply to the ship?

Strategy

We are asked for a force; we know the initial and final speeds (and hence the change in speed), and we know the time interval over which this all happened. In particular, we know the amount of time that the force acted. This suggests using the impulse-momentum relation. To use that, though, we need the mass of the *Enterprise*. An internet search gives a best estimate of the mass of the *Enterprise* (in the 2009 movie) as $2 \times 10^9 \text{ kg}$.

Solution

Because this problem involves only one direction (i.e., the direction of the force applied by the engines), we only need the scalar form of the impulse-momentum theorem **Equation 9.7**, which is

$$\Delta p = J$$

with

$$\Delta p = m\Delta v$$

and

$$J = F\Delta t.$$

Equating these expressions gives

$$F\Delta t = m\Delta v.$$

Solving for the magnitude of the force and inserting the given values leads to

$$F = \frac{m\Delta v}{\Delta t} = \frac{\left(2 \times 10^9 \text{ kg}\right)\left(7.5 \times 10^7 \text{ m/s}\right)}{60 \text{ s}} = 2.5 \times 10^{15} \text{ N}.$$

Significance

This is an unimaginably huge force. It goes almost without saying that such a force would kill everyone on board instantly, as well as destroying every piece of equipment. Fortunately, the *Enterprise* has "inertial dampeners." It is left as an exercise for the reader's imagination to determine how these work.

9.1 **Check Your Understanding** The U.S. Air Force uses "10*g*s" (an acceleration equal to $10 \times 9.8 \text{ m/s}^2$) as the maximum acceleration a human can withstand (but only for several seconds) and survive. How much time must the *Enterprise* spend accelerating if the humans on board are to experience an average of at most 10*g*s of acceleration? (Assume the inertial dampeners are offline.)

Example 9.4

The iPhone Drop

Apple released its iPhone 6 Plus in November 2014. According to many reports, it was originally supposed to have a screen made from sapphire, but that was changed at the last minute for a hardened glass screen. Reportedly, this was because the sapphire screen cracked when the phone was dropped. What force did the iPhone 6 Plus experience as a result of being dropped?

Strategy

The force the phone experiences is due to the impulse applied to it by the floor when the phone collides with the floor. Our strategy then is to use the impulse-momentum relationship. We calculate the impulse, estimate the impact time, and use this to calculate the force.

We need to make a couple of reasonable estimates, as well as find technical data on the phone itself. First, let's suppose that the phone is most often dropped from about chest height on an average-height person. Second, assume that it is dropped from rest, that is, with an initial vertical velocity of zero. Finally, we assume that the phone bounces very little—the height of its bounce is assumed to be negligible.

Solution

Define upward to be the +*y*-direction. A typical height is approximately $h = 1.5 \text{ m}$ and, as stated, $\vec{\mathbf{v}}_i = (0 \text{ m/s})\hat{\mathbf{i}}$. The average force on the phone is related to the impulse the floor applies on it during the collision:

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\vec{\mathbf{J}}}{\Delta t}.$$

The impulse $\vec{\mathbf{J}}$ equals the change in momentum,

$$\vec{\mathbf{J}} = \Delta \vec{\mathbf{p}}$$

so

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t}.$$

Next, the change of momentum is

$$\Delta \vec{\mathbf{p}} = m \Delta \vec{\mathbf{v}}.$$

We need to be careful with the velocities here; this is the change of velocity due to the collision with the floor. But the phone also has an initial drop velocity [$\vec{\mathbf{v}}_{\text{i}} = (0 \text{ m/s}) \hat{\mathbf{j}}$], so we label our velocities. Let:

- $\vec{\mathbf{v}}_{\text{i}} =$ the initial velocity with which the phone was dropped (zero, in this example)
- $\vec{\mathbf{v}}_1 =$ the velocity the phone had the instant just before it hit the floor
- $\vec{\mathbf{v}}_2 =$ the final velocity of the phone as a result of hitting the floor

Figure 9.12 shows the velocities at each of these points in the phone's trajectory.

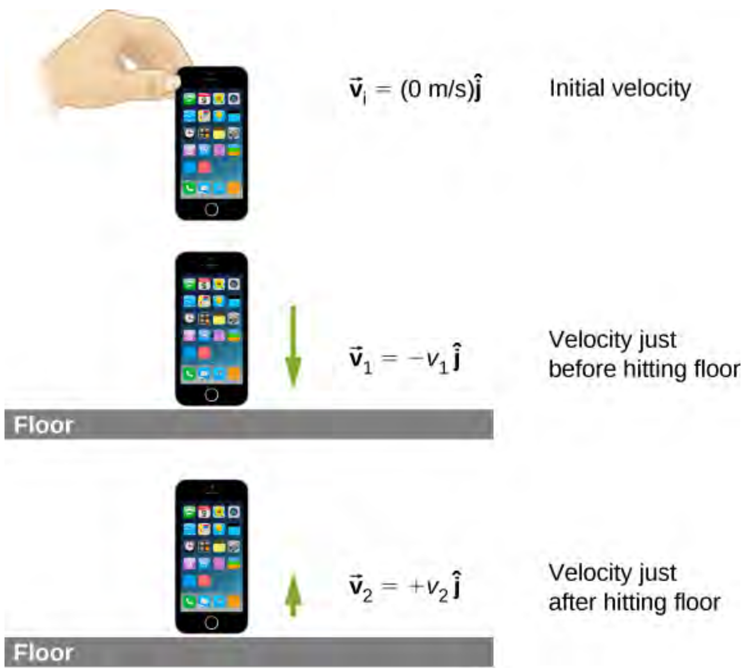


Figure 9.12 (a) The initial velocity of the phone is zero, just after the person drops it. (b) Just before the phone hits the floor, its velocity is $\vec{\mathbf{v}}_1$, which is unknown at the moment, except for its direction, which is downward $(-\hat{\mathbf{j}})$. (c) After bouncing off the floor, the phone has a velocity $\vec{\mathbf{v}}_2$, which is also unknown, except for its direction, which is upward $(+\hat{\mathbf{j}})$.

With these definitions, the change of momentum of the phone during the collision with the floor is

$$m \Delta \vec{\mathbf{v}} = m(\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1).$$

Since we assume the phone doesn't bounce at all when it hits the floor (or at least, the bounce height is negligible), then $\vec{\mathbf{v}}_2$ is zero, so

$$\begin{aligned} m\Delta\vec{\mathbf{v}} &= m\left[0-\left(-v_1\,\hat{\mathbf{j}}\right)\right] \\ m\Delta\vec{\mathbf{v}} &= +mv_1\,\hat{\mathbf{j}}. \end{aligned}$$

We can get the speed of the phone just before it hits the floor using either kinematics or conservation of energy. We'll use conservation of energy here; you should re-do this part of the problem using kinematics and prove that you get the same answer.

First, define the zero of potential energy to be located at the floor. Conservation of energy then gives us:

$$\begin{aligned} E_i &= E_1 \\ K_i + U_i &= K_1 + U_1 \\ \frac{1}{2}mv_i^2 + mgh_{\text{drop}} &= \frac{1}{2}mv_1^2 + mgh_{\text{floo}}. \end{aligned}$$

Defining $h_{\text{floo}} = 0$ and using $\vec{\mathbf{v}}_i = (0\text{ m/s})\,\hat{\mathbf{j}}$ gives

$$\begin{aligned} \frac{1}{2}mv_1^2 &= mgh_{\text{drop}} \\ v_1 &= \pm\sqrt{2gh_{\text{drop}}}. \end{aligned}$$

Because v_1 is a vector magnitude, it must be positive. Thus, $m\Delta v = mv_1 = m\sqrt{2gh_{\text{drop}}}$. Inserting this result into the expression for force gives

$$\begin{aligned} \vec{\mathbf{F}} &= \frac{\Delta\vec{\mathbf{p}}}{\Delta t} \\ &= \frac{m\Delta\vec{\mathbf{v}}}{\Delta t} \\ &= \frac{+mv_1\,\hat{\mathbf{j}}}{\Delta t} \\ &= \frac{m\sqrt{2gh}}{\Delta t}\,\hat{\mathbf{j}}. \end{aligned}$$

Finally, we need to estimate the collision time. One common way to estimate a collision time is to calculate how long the object would take to travel its own length. The phone is moving at 5.4 m/s just before it hits the floor, and it is 0.14 m long, giving an estimated collision time of 0.026 s. Inserting the given numbers, we obtain

$$\vec{\mathbf{F}} = \frac{(0.172\text{ kg})\sqrt{2\left(9.8\text{ m/s}^2\right)(1.5\text{ m})}}{0.026\text{ s}}\,\hat{\mathbf{j}} = (36\text{ N})\,\hat{\mathbf{j}}.$$

Significance

The iPhone itself weighs just $(0.172\text{ kg})(9.81\text{ m/s}^2) = 1.68\text{ N}$; the force the floor applies to it is therefore over 20 times its weight.

9.2 Check Your Understanding What if we had assumed the phone *did* bounce on impact? Would this have increased the force on the iPhone, decreased it, or made no difference?

Momentum and Force

In **Example 9.3**, we obtained an important relationship:

$$\vec{\mathbf{F}}_{\text{ave}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t}. \tag{9.8}$$

In words, the average force applied to an object is equal to the change of the momentum that the force causes, divided by the time interval over which this change of momentum occurs. This relationship is very useful in situations where the collision time Δt is small, but measureable; typical values would be 1/10th of a second, or even one thousandth of a second. Car crashes, punting a football, or collisions of subatomic particles would meet this criterion.

For a *continuously* changing momentum—due to a continuously changing force—this becomes a powerful conceptual tool. In the limit $\Delta t \to dt$, **Equation 9.2** becomes

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}. \tag{9.9}$$

This says that the rate of change of the system's momentum (implying that momentum is a function of time) is exactly equal to the net applied force (also, in general, a function of time). This is, in fact, Newton's second law, written in terms of momentum rather than acceleration. This is the relationship Newton himself presented in his *Principia Mathematica* (although he called it "quantity of motion" rather than "momentum").

If the mass of the system remains constant, **Equation 9.3** reduces to the more familiar form of Newton's second law. We can see this by substituting the definition of momentum:

$$\vec{\mathbf{F}} = \frac{d(m\vec{\mathbf{v}})}{dt} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}.$$

The assumption of constant mass allowed us to pull m out of the derivative. If the mass is not constant, we cannot use this form of the second law, but instead must start from **Equation 9.3**. Thus, one advantage to expressing force in terms of changing momentum is that it allows for the mass of the system to change, as well as the velocity; this is a concept we'll explore when we study the motion of rockets.

Newton's Second Law of Motion in Terms of Momentum

The net external force on a system is equal to the rate of change of the momentum of that system caused by the force:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}.$$

Although **Equation 9.3** allows for changing mass, as we will see in **Rocket Propulsion**, the relationship between momentum and force remains useful when the mass of the system is constant, as in the following example.

Example 9.5

Calculating Force: Venus Williams' Tennis Serve

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet? Assume that the ball's speed just after impact is 58 m/s, as shown in **Figure 9.13**, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms.

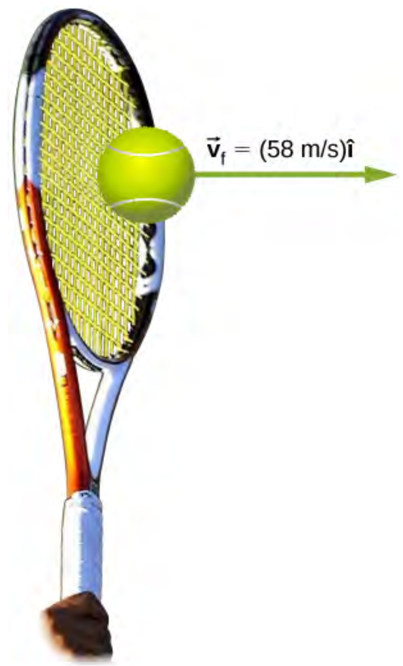


Figure 9.13 The final velocity of the tennis ball is $\vec{\mathbf{v}}_f = (58 \text{ m/s})\hat{\mathbf{i}}$.

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}.$$

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p = m\Delta v = m(v_f - v_i)$$

where we have used scalars because this problem involves only one dimension. In this example, the velocity just after impact and the time interval are given; thus, once Δp is calculated, we can use $F = \frac{\Delta p}{\Delta t}$ to find the force.

Solution

To determine the change in momentum, insert the values for the initial and final velocities into the equation above:

$$\begin{aligned} \Delta p &= m(v_f - v_i) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.3 \frac{\text{kg} \cdot \text{m}}{\text{s}}. \end{aligned}$$

Now the magnitude of the net external force can be determined by using

$$F = \frac{\Delta p}{\Delta t} = \frac{3.3 \frac{\text{kg} \cdot \text{m}}{\text{s}}}{5.0 \times 10^{-3} \text{ s}} = 6.6 \times 10^2 \text{ N}.$$

where we have retained only two significant figures in the final step.

Significance

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.57-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F = ma$, but one additional step would be required compared with the strategy used in this example.

9.3 | Conservation of Linear Momentum

Learning Objectives

By the end of this section, you will be able to:

- Explain the meaning of "conservation of momentum"
- Correctly identify if a system is, or is not, closed
- Define a system whose momentum is conserved
- Mathematically express conservation of momentum for a given system
- Calculate an unknown quantity using conservation of momentum

Recall Newton's third law: When two objects of masses m_1 and m_2 interact (meaning that they apply forces on each other), the force that object 2 applies to object 1 is equal in magnitude and opposite in direction to the force that object 1 applies on object 2. Let:

- $\vec{\mathbf{F}}_{21} =$ the force on m_1 from m_2
- $\vec{\mathbf{F}}_{12} =$ the force on m_2 from m_1

Then, in symbols, Newton's third law says

$$\begin{aligned} \vec{\mathbf{F}}_{21} &= -\vec{\mathbf{F}}_{12} \\ m_1 \vec{\mathbf{a}}_{\mathbf{1}} &= -m_2 \vec{\mathbf{a}}_{\mathbf{2}}. \end{aligned} \tag{9.10}$$

(Recall that these two forces do not cancel because they are applied to different objects. F_{21} causes m_1 to accelerate, and F_{12} causes m_2 to accelerate.)

Although the magnitudes of the forces on the objects are the same, the accelerations are not, simply because the masses (in general) are different. Therefore, the changes in velocity of each object are different:

$$\frac{d\vec{\mathbf{v}}_{\mathbf{1}}}{dt} \neq \frac{d\vec{\mathbf{v}}_{\mathbf{2}}}{dt}.$$

However, the products of the mass and the change of velocity *are* equal (in magnitude):

$$m_1 \frac{d\vec{\mathbf{v}}_{\mathbf{1}}}{dt} = -m_2 \frac{d\vec{\mathbf{v}}_{\mathbf{2}}}{dt}. \tag{9.11}$$

It's a good idea, at this point, to make sure you're clear on the physical meaning of the derivatives in **Equation 9.3**. Because of the interaction, each object ends up getting its velocity changed, by an amount *dv*. Furthermore, the interaction occurs over a time interval *dt*, which means that the change of velocities also occurs over *dt*. This time interval is the same for each object.

Let's assume, for the moment, that the masses of the objects do not change during the interaction. (We'll relax this restriction later.) In that case, we can pull the masses inside the derivatives:

$$\frac{d}{dt}\left(m_1 \vec{\mathbf{v}}_{\mathbf{1}}\right) = -\frac{d}{dt}\left(m_2 \vec{\mathbf{v}}_{\mathbf{2}}\right) \tag{9.12}$$

and thus

$$\frac{d\vec{\mathbf{p}}_1}{dt} = -\frac{d\vec{\mathbf{p}}_2}{dt}. \tag{9.13}$$

This says that *the rate at which momentum changes is the same for both objects*. The masses are different, and the changes of velocity are different, but the rate of change of the product of m and $\vec{\mathbf{v}}$ are the same.

Physically, this means that during the interaction of the two objects (m_1 and m_2), both objects have their momentum changed; but those changes are identical in magnitude, though opposite in sign. For example, the momentum of object 1 might increase, which means that the momentum of object 2 decreases by exactly the same amount.

In light of this, let's re-write **Equation 9.12** in a more suggestive form:

$$\frac{d\vec{\mathbf{p}}_1}{dt} + \frac{d\vec{\mathbf{p}}_2}{dt} = 0. \tag{9.14}$$

This says that during the interaction, although object 1's momentum changes, and object 2's momentum also changes, these two changes cancel each other out, so that the total change of momentum of the two objects together is zero.

Since the total combined momentum of the two objects together never changes, then we could write

$$\frac{d}{dt}\left(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2\right) = 0 \tag{9.15}$$

from which it follows that

$$\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = \text{constant}. \tag{9.16}$$

As shown in **Figure 9.14**, the total momentum of the system before and after the collision remains the same.

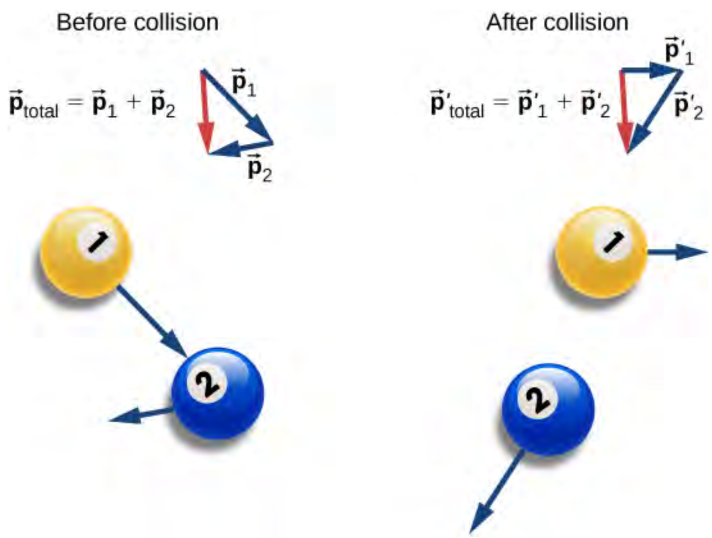


Figure 9.14 Before the collision, the two billiard balls travel with momenta $\vec{p}_1$ and $\vec{p}_3$. The total momentum of the system is the sum of these, as shown by the red vector labeled $\vec{p}_{total}$ on the left. After the collision, the two billiard balls travel with different momenta $\vec{p}'_1$ and $\vec{p}'_3$. The total momentum, however, has not changed, as shown by the red vector arrow $\vec{p}'_{total}$ on the right.

Generalizing this result to *N* objects, we obtain

$$\begin{aligned} \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N &= \text{constant} \\ \sum_{j=1}^{N} \vec{p}_j &= \text{constant.} \end{aligned} \quad (9.17)$$

Equation 9.17 is the definition of the total (or net) momentum of a system of *N* interacting objects, along with the statement that the total momentum of a system of objects is constant in time—or better, is conserved.

Conservation Laws

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. *The mass of the system must remain constant during the interaction.*
   As the objects interact (apply forces on each other), they may *transfer* mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes:
   $$\left[\frac{dm}{dt}\right]_{\text{system}} = 0.$$
2. *The net external force on the system must be zero.*
   As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in

magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{\mathbf{F}}_{\text{ext}} = \vec{\mathbf{0}}.$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Thus, the more compact way to express this is shown below.

Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j=1}^{N} \vec{\mathbf{p}}_{\mathbf{j}} = \text{constant}.$$

This statement is called the **Law of Conservation of Momentum**. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. *In a closed system, the total momentum never changes.*

Note that there absolutely *can* be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the *net* external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no *net* force.

The Meaning of 'System'

A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (**Figure 9.15**).

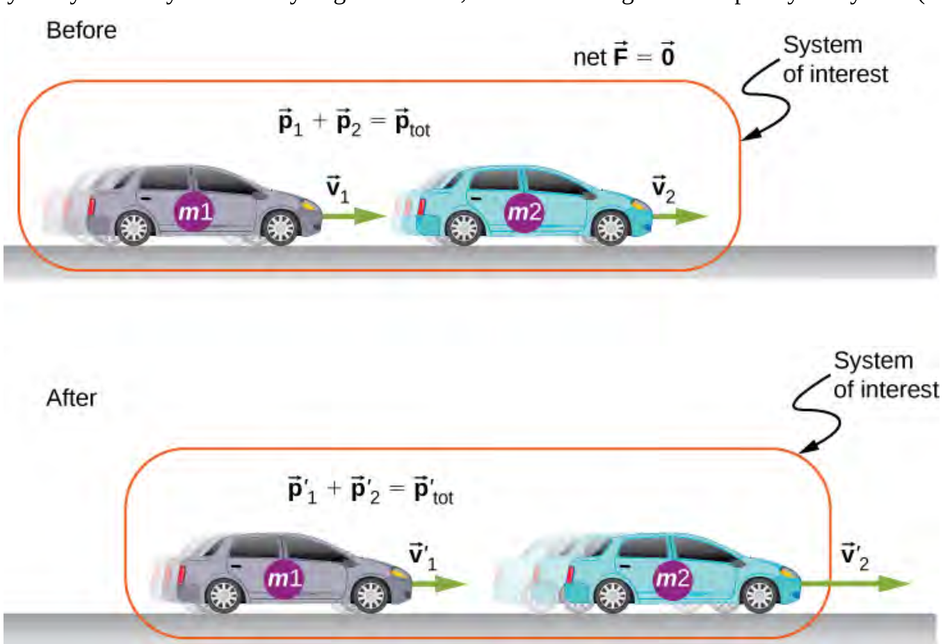


Figure 9.15 The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.

Problem-Solving Strategy: Conservation of Momentum

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the “event” (explosion or collision).
3. Write down an expression representing the total momentum of the system after the “event.”
4. Set these two expressions equal to each other, and solve this equation for the desired quantity.

Example 9.6

Colliding Carts

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure 9.16). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?

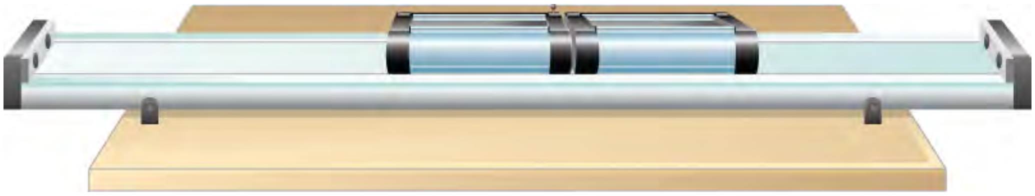


Figure 9.16 Two lab carts collide and stick together after the collision.

Strategy

We have a collision. We’re given masses and initial velocities; we’re asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn’t change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

Solution

Conservation of momentum is

$$\vec{\mathbf{p}}_{\mathbf{f}} = \vec{\mathbf{p}}_{\mathbf{i}}.$$

Define the direction of their initial velocity vectors to be the +x-direction. The initial momentum is then

$$\vec{\mathbf{p}}_{\mathbf{i}} = m_1 v_1 \hat{\mathbf{i}} + m_2 v_2 \hat{\mathbf{i}}.$$

The final momentum of the now-linked carts is

$$\vec{\mathbf{p}}_{\mathbf{f}} = (m_1 + m_2)\vec{\mathbf{v}}_{\mathbf{f}}.$$

Equating:

$$\begin{aligned}(m_1 + m_2)\vec{\mathbf{v}}_{\mathbf{f}} &= m_1 v_1 \hat{\mathbf{i}} + m_2 v_2 \hat{\mathbf{i}} \\ \vec{\mathbf{v}}_{\mathbf{f}} &= \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)\hat{\mathbf{i}}.\end{aligned}$$

Substituting the given numbers:

$$\begin{aligned}\vec{\mathbf{v}}_{\mathbf{f}} &= \left[\frac{(0.675\ \text{kg})(0.75\ \text{m/s}) + (0.5\ \text{kg})(1.33\ \text{m/s})}{1.175\ \text{kg}}\right]\hat{\mathbf{i}} \\ &= (0.997\ \text{m/s})\,\hat{\mathbf{i}}\,.\end{aligned}$$

Significance

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

9.3 Check Your Understanding Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

Example 9.7

A Bouncing Superball

A superball of mass 0.25 kg is dropped from rest from a height of $h = 1.50\ \text{m}$ above the floor. It bounces with no loss of energy and returns to its initial height (Figure 9.17).

a. What is the superball's change of momentum during its bounce on the floor?

b. What was Earth's change of momentum due to the ball colliding with the floor?

c. What was Earth's change of velocity as a result of this collision?

(This example shows that you have to be careful about defining your system.)

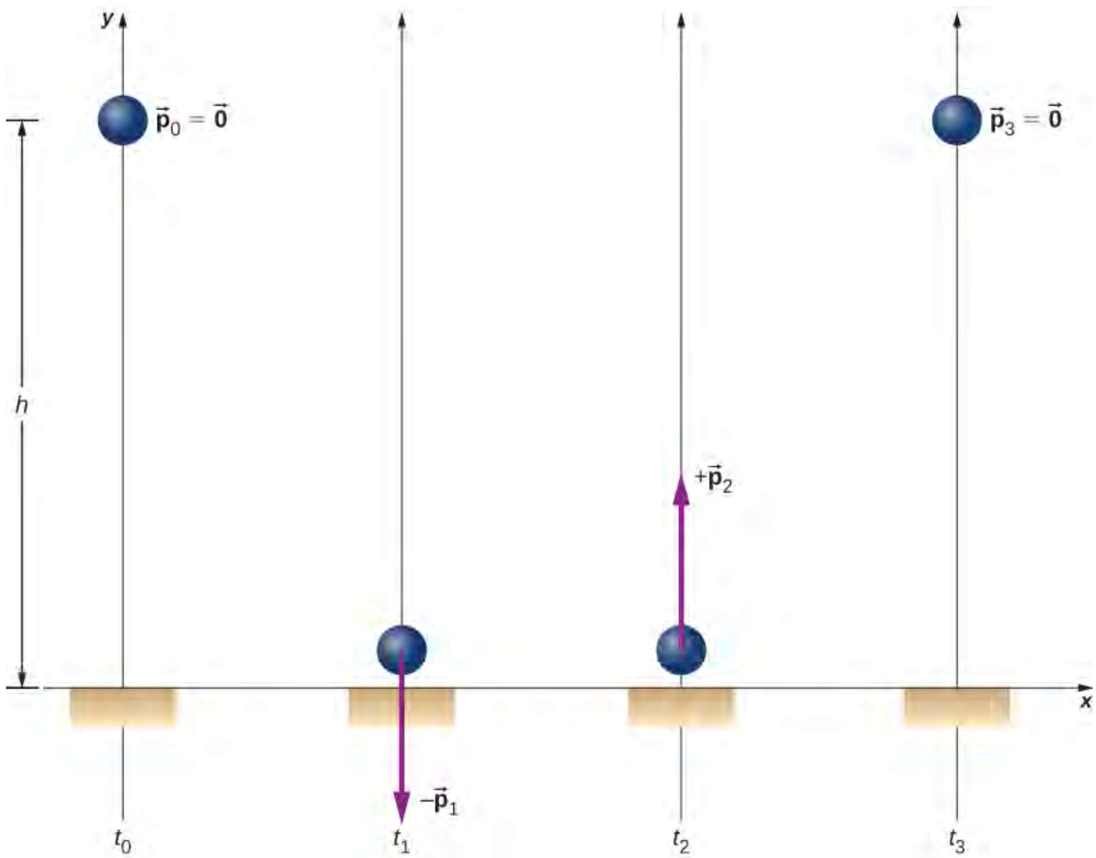


Figure 9.17 A superball is dropped to the floor (t_0), hits the floor (t_1), bounces (t_2), and returns to its initial height (t_3).

Strategy

Since we are asked only about the ball's change of momentum, we define our system to be the ball. But this is clearly not a closed system; gravity applies a downward force on the ball while it is falling, and the normal force from the floor applies a force during the bounce. Thus, we cannot use conservation of momentum as a strategy. Instead, we simply determine the ball's momentum just before it collides with the floor and just after, and calculate the difference. We have the ball's mass, so we need its velocities.

Solution

a. Since this is a one-dimensional problem, we use the scalar form of the equations. Let:

- $p_0 =$ the magnitude of the ball's momentum at time t_0, the moment it was released; since it was dropped from rest, this is zero.
- $p_1 =$ the magnitude of the ball's momentum at time t_1, the instant just before it hits the floor.
- $p_2 =$ the magnitude of the ball's momentum at time t_2, just after it loses contact with the floor after the bounce.

The ball's change of momentum is

$$\begin{aligned}
\Delta \vec{\mathbf{p}} &= \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \\
&= p_2 \hat{\mathbf{j}} - \left(-p_1 \hat{\mathbf{j}}\right) \\
&= (p_2 + p_1)\hat{\mathbf{j}}.
\end{aligned}$$

Its velocity just before it hits the floor can be determined from either conservation of energy or kinematics. We use kinematics here; you should re-solve it using conservation of energy and confirm you get the same result.

We want the velocity just before it hits the ground (at time t_1). We know its initial velocity $v_0 = 0$ (at time t_0), the height it falls, and its acceleration; we don't know the fall time. We could calculate that, but instead we use

$$\vec{\mathbf{v}}_1 = -\hat{\mathbf{j}}\sqrt{2gy} = -5.4 \text{ m/s}\,\hat{\mathbf{j}}.$$

Thus the ball has a momentum of

$$\begin{aligned}
\vec{\mathbf{p}}_1 &= -(0.25 \text{ kg})\left(-5.4 \text{ m/s}\,\hat{\mathbf{j}}\right) \\
&= -(1.4 \text{ kg} \cdot \text{m/s})\hat{\mathbf{j}}.
\end{aligned}$$

We don't have an easy way to calculate the momentum after the bounce. Instead, we reason from the symmetry of the situation.

Before the bounce, the ball starts with zero velocity and falls 1.50 m under the influence of gravity, achieving some amount of momentum just before it hits the ground. On the return trip (after the bounce), it starts with some amount of momentum, rises the same 1.50 m it fell, and ends with zero velocity. Thus, the motion after the bounce was the mirror image of the motion before the bounce. From this symmetry, it must be true that the ball's momentum after the bounce must be equal and opposite to its momentum before the bounce. (This is a subtle but crucial argument; make sure you understand it before you go on.) Therefore,

$$\vec{\mathbf{p}}_2 = -\vec{\mathbf{p}}_1 = +(1.4 \text{ kg} \cdot \text{m/s})\hat{\mathbf{j}}.$$

Thus, the ball's change of velocity during the bounce is

$$\begin{aligned}
\Delta \vec{\mathbf{p}} &= \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \\
&= (1.4 \text{ kg} \cdot \text{m/s})\hat{\mathbf{j}} - (-1.4 \text{ kg} \cdot \text{m/s})\hat{\mathbf{j}} \\
&= +(2.8 \text{ kg} \cdot \text{m/s})\hat{\mathbf{j}}.
\end{aligned}$$

b. What was Earth's change of momentum due to the ball colliding with the floor?
Your instinctive response may well have been either "zero; the Earth is just too massive for that tiny ball to have affected it" or possibly, "more than zero, but utterly negligible." But no—if we re-define our system to be the Superball + Earth, then this system is closed (neglecting the gravitational pulls of the Sun, the Moon, and the other planets in the solar system), and therefore the total change of momentum of this new system must be zero. Therefore, Earth's change of momentum is exactly the same magnitude:

$$\Delta \vec{\mathbf{p}}_{\text{Earth}} = -2.8 \text{ kg} \cdot \text{m/s}\,\hat{\mathbf{j}}.$$

c. What was Earth's change of velocity as a result of this collision?
This is where your instinctive feeling is probably correct:

$$\begin{aligned}\Delta \vec{\mathbf{v}}_{\text{Earth}} &= \frac{\Delta \vec{\mathbf{p}}_{\text{Earth}}}{M_{\text{Earth}}} \\ &= -\frac{2.8\ \text{kg}\cdot\text{m/s}}{5.97\times 10^{24}\ \text{kg}}\hat{\mathbf{j}} \\ &= -\left(4.7\times 10^{-25}\ \text{m/s}\right)\hat{\mathbf{j}}\,.\end{aligned}$$

This change of Earth's velocity *is* utterly negligible.

Significance

It is important to realize that the answer to part (c) is not a velocity; it is a change of velocity, which is a very different thing. Nevertheless, to give you a feel for just how small that change of velocity is, suppose you were moving with a velocity of 4.7×10^{-25} m/s . At this speed, it would take you about 7 million years to travel a distance equal to the diameter of a hydrogen atom.

9.4 Check Your Understanding Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

Example 9.8

Ice Hockey 1

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 9.18). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

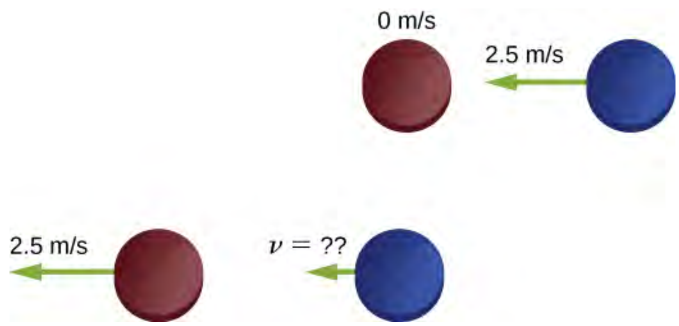


Figure 9.18 Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

- zero

- 2.5 m/s to the left
- 2.5 m/s to the right
- 1.25 m/s to the left
- 1.25 m/s to the right
- something else

Solution

Define the +*x*-direction to point to the right. Conservation of momentum then reads

$$\begin{aligned} \vec{\mathbf{p}}_{\mathbf{f}} &= \vec{\mathbf{p}}_{\mathbf{i}} \\ mv_{r_f}\hat{\mathbf{i}} + mv_{b_f}\hat{\mathbf{i}} &= mv_{r_i}\hat{\mathbf{i}} - mv_{b_i}\hat{\mathbf{i}}. \end{aligned}$$

Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$\begin{aligned} mv_{r_f}\hat{\mathbf{i}} + mv_{b_f}\hat{\mathbf{i}} &= -mv_{b_i}\hat{\mathbf{i}} \\ v_{r_f}\hat{\mathbf{i}} + v_{b_f}\hat{\mathbf{i}} &= -v_{b_i}\hat{\mathbf{i}}. \end{aligned}$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$\begin{aligned} -(2.5\text{ m/s})\hat{\mathbf{i}} + \vec{\mathbf{v}}_{b_f} &= -(2.5\text{ m/s})\hat{\mathbf{i}} \\ \vec{\mathbf{v}}_{b_f} &= 0. \end{aligned}$$

Significance

Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

9.5 Check Your Understanding Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

Example 9.9

Landing of *Philae*

On November 12, 2014, the European Space Agency successfully landed a probe named *Philae* on Comet 67P/Churyumov/Gerasimenko (**Figure 9.19**). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.

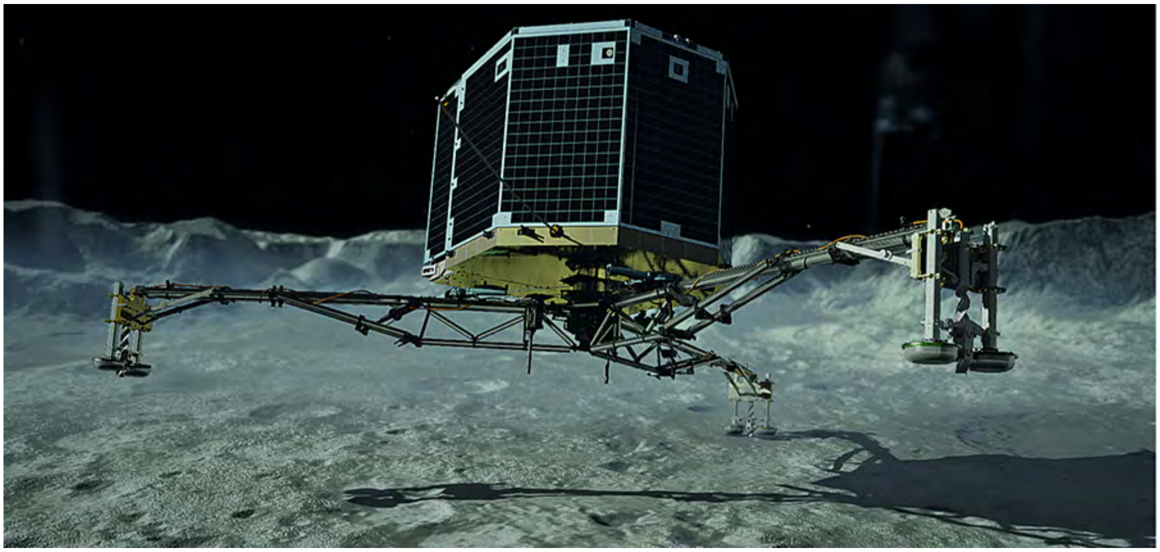


Figure 9.19 An artist's rendering of *Philae* landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the +*y*-direction, perpendicular to the surface of the comet, and $y = 0$ to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13}$ kg
- The acceleration due to the comet's gravity: $\vec{\mathbf{a}} = -\left(5.0 \times 10^{-3}\ \text{m/s}^2\right)\hat{\mathbf{j}}$
- *Philae's* mass: $M_p = 96$ kg
- Initial touchdown speed: $\vec{\mathbf{v}}_1 = -(1.0\ \text{m/s})\,\hat{\mathbf{j}}$
- Initial upward speed due to first bounce: $\vec{\mathbf{v}}_2 = (0.38\ \text{m/s})\,\hat{\mathbf{j}}$
- Landing impact time: $\Delta t = 1.3$ s

Strategy

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we *are* told that the *Philae* lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both *Philae* and Comet 67/P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

Solution

Let $\vec{\mathbf{p}}_1$ be *Philae's* momentum at the moment just before touchdown, and $\vec{\mathbf{p}}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\vec{\mathbf{p}}_1 = M_p\,\vec{\mathbf{v}}_1 = (96\ \text{kg})\left(-1.0\ \text{m/s}\,\hat{\mathbf{j}}\right) = -(96\ \text{kg}\cdot\text{m/s})\,\hat{\mathbf{j}}$$

and just after was

$$\vec{\mathbf{p}}_2 = M_p \vec{\mathbf{v}}_2 = (96\text{ kg})\left(+0.38\text{ m/s}\,\hat{\mathbf{j}}\right) = (36.5\text{ kg}\cdot\text{m/s})\,\hat{\mathbf{j}}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta\vec{\mathbf{p}} &= \vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 \\ &= (36.5\text{ kg}\cdot\text{m/s})\,\hat{\mathbf{j}} - \left(-96.0\text{ kg}\cdot\text{m/s}\,\hat{\mathbf{j}}\right) = (133\text{ kg}\cdot\text{m/s})\,\hat{\mathbf{j}}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the *comet's* momentum changed by exactly the negative of this:

$$\Delta\vec{\mathbf{p}}_c = -\Delta\vec{\mathbf{p}} = -(133\text{ kg}\cdot\text{m/s})\,\hat{\mathbf{j}}.$$

Therefore, its change of velocity is

$$\Delta\vec{\mathbf{v}}_c = \frac{\Delta\vec{\mathbf{p}}_c}{M_c} = \frac{-(133\text{ kg}\cdot\text{m/s})\,\hat{\mathbf{j}}}{1.0\times10^{13}\text{ kg}} = -\left(1.33\times10^{-11}\text{ m/s}\right)\hat{\mathbf{j}}.$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is *not* zero.

9.6 Check Your Understanding The changes of momentum for *Philae* and for Comet 67/P were equal (in magnitude). Were the impulses experienced by *Philae* and the comet equal? How about the forces? How about the changes of kinetic energies?

9.4 | Types of Collisions

Learning Objectives

By the end of this section, you will be able to:

- Identify the type of collision
- Correctly label a collision as elastic or inelastic
- Use kinetic energy along with momentum and impulse to analyze a collision

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (one-to-many).
- Multiple objects can collide and stick together, forming a single object (many-to-one).
- Multiple objects can collide and bounce off of each other, remaining as multiple objects (many-to-many). If they do bounce off each other, then they may recoil at the same speeds with which they approached each other before the collision, or they may move off more slowly.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

One-to-Many

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system *will* have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

Many-to-One

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called **inelastic**.

In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; the loss of kinetic energy is a maximum. Such a collision is said to be **perfectly inelastic**.

Many-to-Many

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called **elastic**.

In any interaction of a closed system of objects, the total momentum of the system is conserved ($\vec{\mathbf{p}}_f = \vec{\mathbf{p}}_\text{i}$) but the kinetic energy may not be:

- If $0 < K_\text{f} < K_\text{i}$, the collision is inelastic.
- If $K_\text{f} = 0$, the collision is perfectly inelastic.
- If $K_\text{f} = K_\text{i}$, the collision is elastic.
- If $K_\text{f} > K_\text{i}$, the interaction is an explosion.

The point of all this is that, in analyzing a collision or explosion, you can use both momentum and kinetic energy.

Problem-Solving Strategy: Collisions

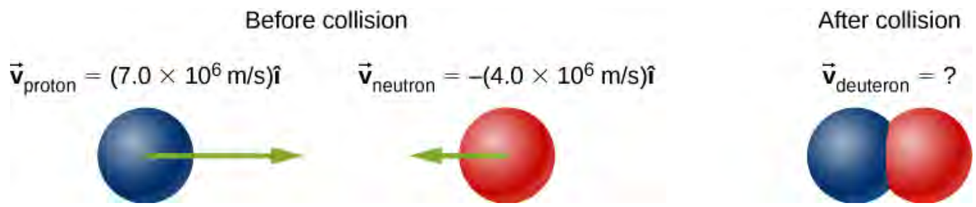
A closed system always conserves momentum; it might also conserve kinetic energy, but very often it doesn't. Energy-momentum problems confined to a plane (as ours are) usually have two unknowns. Generally, this approach works well:

1. Define a closed system.
2. Write down the expression for conservation of momentum.
3. If kinetic energy is conserved, write down the expression for conservation of kinetic energy; if not, write down the expression for the change of kinetic energy.
4. You now have two equations in two unknowns, which you solve by standard methods.

Example 9.10

Formation of a Deuteron

A proton (mass 1.67×10^{-27} kg) collides with a neutron (with essentially the same mass as the proton) to form a particle called a *deuteron*. What is the velocity of the deuteron if it is formed from a proton moving with velocity 7.0×10^6 m/s to the left and a neutron moving with velocity 4.0×10^6 m/s to the right?



Strategy

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of energy to determine the final velocity of the system.

Solution

Treat the two particles as having identical masses M. Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d.$$

The masses divide out:

$$\begin{aligned} v_p - v_n &= 2v_d \\ 7.0 \times 10^6 \text{ m/s} - 4.0 \times 10^6 \text{ m/s} &= 2v_d \\ v_d &= 1.5 \times 10^6 \text{ m/s}. \end{aligned}$$

The velocity is thus $\vec{v}_d = \left(1.5 \times 10^6 \text{ m/s}\right)\hat{i}$.

Significance

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called “daughter particles.”

Example 9.11

Ice Hockey 2

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (**Figure 9.20**). If the collision is perfectly elastic, what are the final velocities of the two pucks?

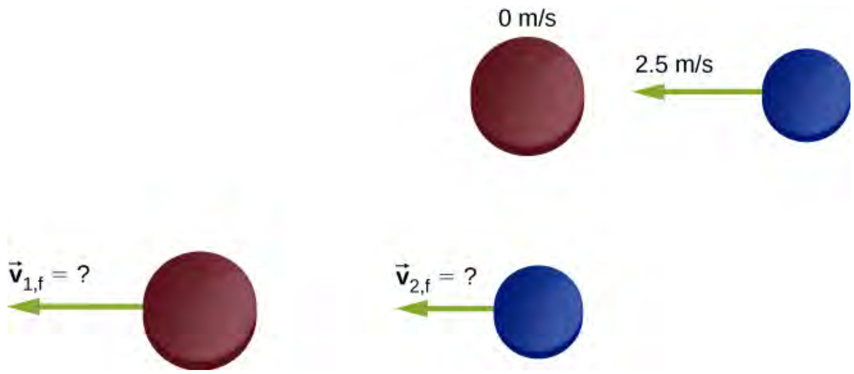


Figure 9.20 Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, and we're told their masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

Solution

Conservation of momentum, in this case, reads

$$\begin{aligned} p_i &= p_f \\ m_2 v_{2,i} &= m_1 v_{1,f} + m_2 v_{2,f}. \end{aligned}$$

Conservation of kinetic energy reads

$$\begin{aligned} K_i &= K_f \\ \frac{1}{2} m_2 v_{2,i}^2 &= \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \end{aligned}$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$\begin{aligned} v_{1,f} &= \frac{(m_1 - m_2)v_{1,i} + 2m_2 v_{2,i}}{m_1 + m_2} \\ v_{2f} &= \frac{(m_2 - m_1)v_{2,i} + 2m_1 v_{1,i}}{m_1 + m_2}. \end{aligned}$$

Substituting the given numbers, we obtain

$$\begin{aligned} v_{1,f} &= 2.22 \frac{\text{m}}{\text{s}} \\ v_{2,f} &= -0.28 \frac{\text{m}}{\text{s}}. \end{aligned}$$

Significance

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.

9.7 Check Your Understanding There is a second solution to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,\text{f}} = -2.5\text{ m/s}$, $v_{2,\text{f}} = 0$. This solution is unacceptable on physical grounds; what's wrong with it?

Example 9.12

Thor vs. Iron Man

The 2012 movie "The Avengers" has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

a. Estimate the mass of Thor's hammer
b. Estimate how much kinetic energy was lost in this collision

Strategy

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

Solution

a. First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:

   - $M_\text{H} =$ mass of the hammer
   - $M_\text{I} =$ mass of Iron Man
   - $v_\text{H} =$ velocity of the hammer before hitting Iron Man
   - $v =$ combined velocity of Iron Man + hammer after the collision

   Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_\text{H} v_\text{H} = (M_\text{H} + M_\text{I})v.$$

   We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_\text{H} v_\text{H} &= M_\text{H} v + M_\text{I} v \\ M_\text{H}(v_\text{H} - v) &= M_\text{I} v \\ M_\text{H} &= \frac{M_\text{I} v}{v_\text{H} - v} \\ &= \frac{(200\text{ kg})\left(\frac{2\text{ m}}{0.75\text{ s}}\right)}{10\,\frac{\text{m}}{\text{s}} - \left(\frac{2\text{ m}}{0.75\text{ s}}\right)} \\ &= 73\text{ kg}. \end{aligned}$$

   Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_\text{H} = 7 \times 10^1\text{ kg}$.

b. The initial kinetic energy of the system, like the initial momentum, is all in the hammer:

$$\begin{aligned} K_i &= \frac{1}{2}M_H v_H^2 \\ &= \frac{1}{2}(70\text{ kg})(10\text{ m/s})^2 \\ &= 3500\text{ J.} \end{aligned}$$

After the collision,

$$\begin{aligned} K_f &= \frac{1}{2}(M_H + M_I)v^2 \\ &= \frac{1}{2}(70\text{ kg} + 200\text{ kg})(2.67\text{ m/s})^2 \\ &= 960\text{ J.} \end{aligned}$$

Thus, there was a loss of $3500\text{ J} - 960\text{ J} = 2540\text{ J}$.

Significance

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

Example 9.13

Analyzing a Car Crash

At a stoplight, a large truck (3000 kg) collides with a motionless small car (1200 kg). The truck comes to an instantaneous stop; the car slides straight ahead, coming to a stop after sliding 10 meters. The measured coefficient of friction between the car's tires and the road was 0.62. How fast was the truck moving at the moment of impact?

Strategy

At first it may seem we don't have enough information to solve this problem. Although we know the initial speed of the car, we don't know the speed of the truck (indeed, that's what we're asked to find), so we don't know the initial momentum of the system. Similarly, we know the final speed of the truck, but not the speed of the car immediately after impact. The fact that the car eventually slid to a speed of zero doesn't help with the final momentum, since an external friction force caused that. Nor can we calculate an impulse, since we don't know the collision time, or the amount of time the car slid before stopping. A useful strategy is to impose a restriction on the analysis.

Suppose we define a system consisting of just the truck and the car. The momentum of this system isn't conserved, because of the friction between the car and the road. But if we *could* find the speed of the car the instant after impact—before friction had any measurable effect on the car—then we could consider the momentum of the system to be conserved, with that restriction.

Can we find the final speed of the car? Yes; we invoke the work-kinetic energy theorem.

Solution

First, define some variables. Let:

- M_c and M_T be the masses of the car and truck, respectively
- $v_{T,i}$ and$v_{T,f}$ be the velocities of the truck before and after the collision, respectively
- $v_{c,i}$ and$v_{c,f}$ Z be the velocities of the car before and after the collision, respectively
- K_i and K_f be the kinetic energies of the car immediately after the collision, and after the car has stopped sliding (so $K_f = 0$).

- d be the distance the car slides after the collision before eventually coming to a stop.

Since we actually want the initial speed of the truck, and since the truck is not part of the work-energy calculation, let's start with conservation of momentum. For the car + truck system, conservation of momentum reads

$$\begin{aligned} p_\text{i} &= p_\text{f} \\ M_\text{c}\, v_\text{c,i} + M_\text{T}\, v_\text{T,i} &= M_\text{c}\, v_\text{c,f} + M_\text{T}\, v_\text{T,f}. \end{aligned}$$

Since the car's initial velocity was zero, as was the truck's final velocity, this simplifies to

$$v_\text{T,i} = \frac{M_\text{c}}{M_\text{T}} v_\text{c,f}.$$

So now we need the car's speed immediately after impact. Recall that

$$W = \Delta K$$

where

$$\begin{aligned} \Delta K &= K_\text{f} - K_\text{i} \\ &= 0 - \frac{1}{2} M_\text{c}\, v_\text{c,f}^2. \end{aligned}$$

Also,

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = Fd\cos\theta.$$

The work is done over the distance the car slides, which we've called d. Equating:

$$Fd\cos\theta = -\frac{1}{2} M_\text{c}\, v_\text{c,f}^2.$$

Friction is the force on the car that does the work to stop the sliding. With a level road, the friction force is

$$F = \mu_\text{k} M_\text{c}\, g.$$

Since the angle between the directions of the friction force vector and the displacement d is $180°$, and $\cos(180°) = -1,$ we have

$$-(\mu_\text{k} M_\text{c}\, g)d = -\frac{1}{2} M_\text{c}\, v_\text{c,f}^2$$

(Notice that the car's mass divides out; evidently the mass of the car doesn't matter.)

Solving for the car's speed immediately after the collision gives

$$v_\text{c,f} = \sqrt{2\mu_\text{k}\, gd}.$$

Substituting the given numbers:

$$\begin{aligned} v_\text{c,f} &= \sqrt{2(0.62)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(10\text{ m})}. \\ &= 11.0\text{ m/s}. \end{aligned}$$

Now we can calculate the initial speed of the truck:

$$v_\text{T,i} = \left(\frac{1200\text{ kg}}{3000\text{ kg}}\right)\left(11.0\,\frac{\text{m}}{\text{s}}\right) = 4.4\text{ m/s}.$$

Significance

This is an example of the type of analysis done by investigators of major car accidents. A great deal of legal and financial consequences depend on an accurate analysis and calculation of momentum and energy.

9.8 Check Your Understanding Suppose there had been no friction (the collision happened on ice); that would make μ_k zero, and thus $v_{c,f} = \sqrt{2\mu_k gd} = 0$, which is obviously wrong. What is the mistake in this conclusion?

Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of high-energy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What *actually* happened is that nearly *none* of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to $180°$—those alpha-particles reversed direction completely (**Figure 9.21**). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

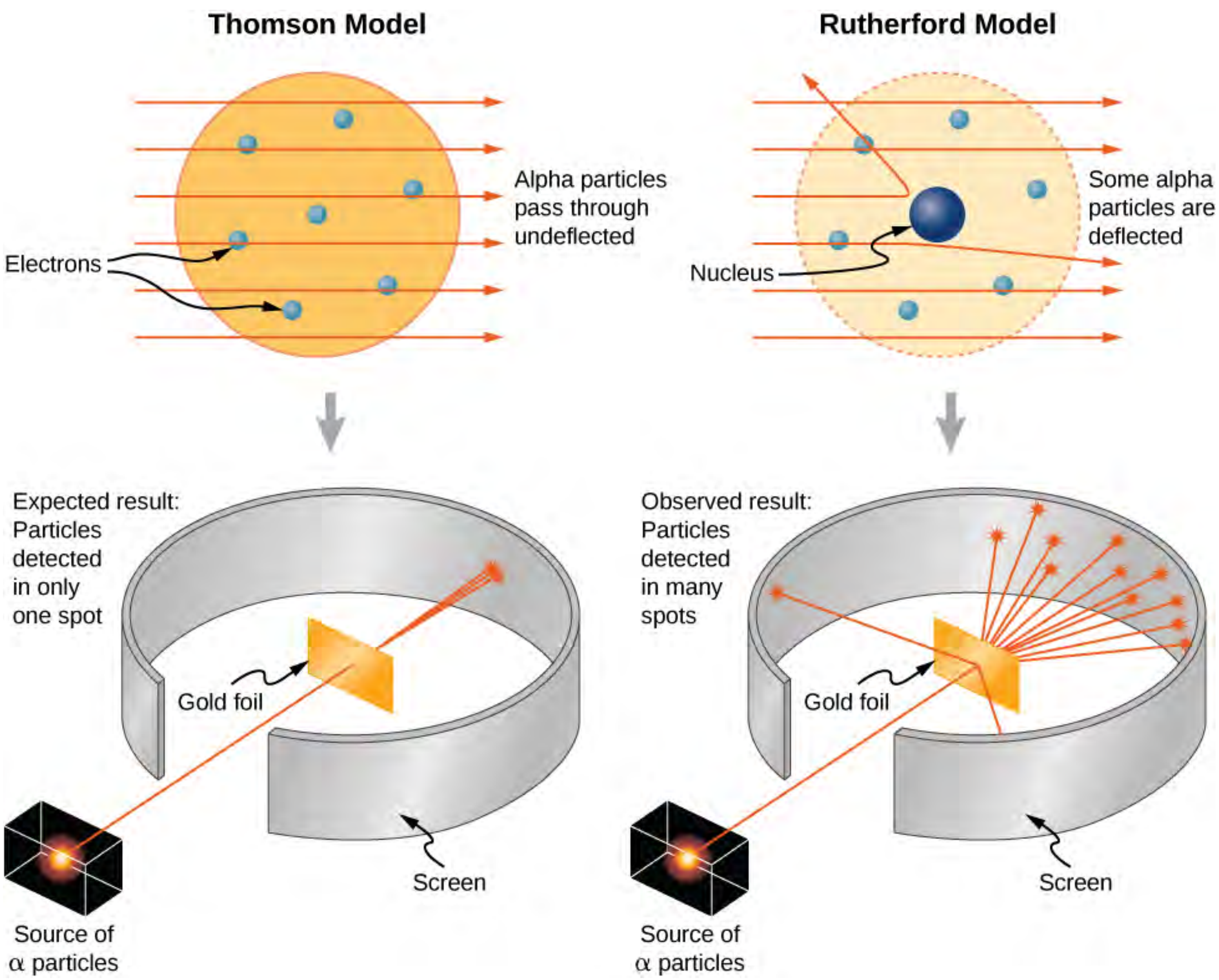


Figure 9.21 The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.

9.5 | Collisions in Multiple Dimensions

Learning Objectives

By the end of this section, you will be able to:

- Express momentum as a two-dimensional vector
- Write equations for momentum conservation in component form
- Calculate momentum in two dimensions, as a vector quantity

It is far more common for collisions to occur in two dimensions; that is, the angle between the initial velocity vectors is neither zero nor $180°$. Let's see what complications arise from this.

The first idea we need is that momentum is a vector; like all vectors, it can be expressed as a sum of perpendicular components (usually, though not always, an *x*-component and a *y*-component, and a *z*-component if necessary). Thus, when we write down the statement of conservation of momentum for a problem, our momentum vectors can be, and usually will be, expressed in component form.

The second idea we need comes from the fact that momentum is related to force:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}.$$

Expressing both the force and the momentum in component form,

$$F_x = \frac{dp_x}{dt}, \quad F_y = \frac{dp_y}{dt}, \quad F_z = \frac{dp_z}{dt}.$$

Remember, these equations are simply Newton's second law, in vector form and in component form. We know that Newton's second law is true in each direction, independently of the others. It follows therefore (via Newton's third law) that conservation of momentum is also true in each direction independently.

These two ideas motivate the solution to two-dimensional problems: We write down the expression for conservation of momentum twice: once in the x-direction and once in the y-direction.

$$\begin{aligned} p_{\text{f},\, x} &= p_{1,\text{i},\, x} + p_{2,\text{i},\, x} \\ p_{\text{f},\, y} &= p_{1,\text{i},\, y} + p_{2,\text{i},\, y} \end{aligned} \tag{9.18}$$

This procedure is shown graphically in **Figure 9.22**.

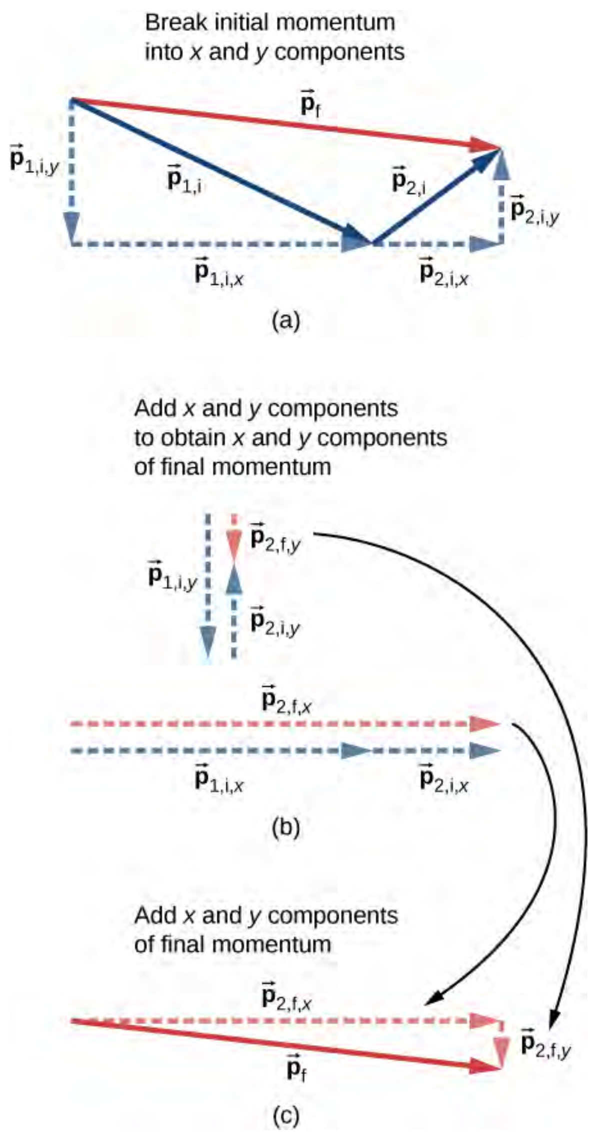


Figure 9.22 (a) For two-dimensional momentum problems, break the initial momentum vectors into their x- and y-components. (b) Add the x- and y-components together separately. This gives you the x- and y-components of the final momentum, which are shown as red dashed vectors. (c) Adding these components together gives the final momentum.

We solve each of these two component equations independently to obtain the x- and y-components of the desired velocity vector:

$$\begin{aligned} v_{f,x} &= \frac{m_1 v_{1,i,x} + m_2 v_{2,i,x}}{m} \\ v_{f,y} &= \frac{m_1 v_{1,i,y} + m_2 v_{2,i,y}}{m}. \end{aligned}$$

(Here, m represents the total mass of the system.) Finally, combine these components using the Pythagorean theorem,

$$v_f = |\vec{v}_f| = \sqrt{v_{f,x}^2 + v_{f,y}^2}.$$

Problem-Solving Strategy: Conservation of Momentum in Two Dimensions

The method for solving a two-dimensional (or even three-dimensional) conservation of momentum problem is generally the same as the method for solving a one-dimensional problem, except that you have to conserve momentum in both (or all three) dimensions simultaneously:

1. Identify a closed system.
2. Write down the equation that represents conservation of momentum in the *x*-direction, and solve it for the desired quantity. If you are calculating a vector quantity (velocity, usually), this will give you the *x*-component of the vector.
3. Write down the equation that represents conservation of momentum in the *y*-direction, and solve. This will give you the *y*-component of your vector quantity.
4. Assuming you are calculating a vector quantity, use the Pythagorean theorem to calculate its magnitude, using the results of steps 3 and 4.

Example 9.14

Traffic Collision

A small car of mass 1200 kg traveling east at 60 km/hr collides at an intersection with a truck of mass 3000 kg that is traveling due north at 40 km/hr (Figure 9.23). The two vehicles are locked together. What is the velocity of the combined wreckage?

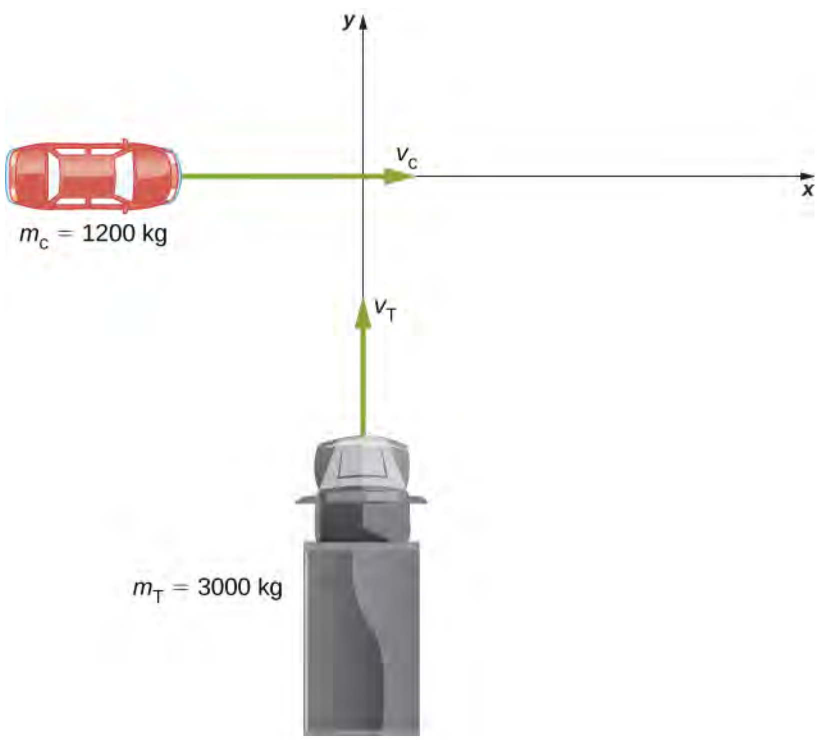


Figure 9.23 A large truck moving north is about to collide with a small car moving east. The final momentum vector has both *x*- and *y*-components.

Strategy

First off, we need a closed system. The natural system to choose is the (car + truck), but this system is not closed; friction from the road acts on both vehicles. We avoid this problem by restricting the question to finding the velocity at the instant just after the collision, so that friction has not yet had any effect on the system. With that restriction, momentum is conserved for this system.

Since there are two directions involved, we do conservation of momentum twice: once in the *x*-direction and once in the *y*-direction.

Solution

Before the collision the total momentum is

$$\vec{\mathbf{p}} = m_c \vec{\mathbf{v}}_c + m_T \vec{\mathbf{v}}_T.$$

After the collision, the wreckage has momentum

$$\vec{\mathbf{p}} = (m_c + m_T) \vec{\mathbf{v}}_w.$$

Since the system is closed, momentum must be conserved, so we have

$$m_c \vec{\mathbf{v}}_c + m_T \vec{\mathbf{v}}_T = (m_c + m_T) \vec{\mathbf{v}}_w.$$

We have to be careful; the two initial momenta are not parallel. We must add vectorially (**Figure 9.24**).

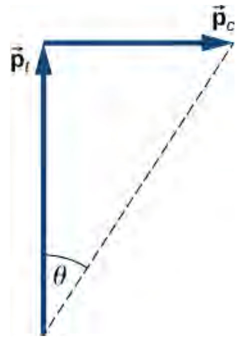


Figure 9.24 Graphical addition of momentum vectors. Notice that, although the car's velocity is larger than the truck's, its momentum is smaller.

If we define the +*x*-direction to point east and the +*y*-direction to point north, as in the figure, then (conveniently),

$$\begin{aligned} \vec{\mathbf{p}}_c &= p_c \hat{\mathbf{i}} = m_c v_c \hat{\mathbf{i}} \\ \vec{\mathbf{p}}_T &= p_T \hat{\mathbf{j}} = m_T v_T \hat{\mathbf{j}}. \end{aligned}$$

Therefore, in the *x*-direction:

$$\begin{aligned} m_c v_c &= (m_c + m_T) v_{w,\, x} \\ v_{w,\, x} &= \left(\frac{m_c}{m_c + m_T}\right) v_c \end{aligned}$$

and in the *y*-direction:

$$\begin{aligned} m_T v_T &= (m_c + m_T) v_{w,\, y} \\ v_{w,\, y} &= \left(\frac{m_T}{m_c + m_T}\right) v_T. \end{aligned}$$

Applying the Pythagorean theorem gives

$$
\begin{aligned}
|\vec{\mathbf{v}}_w| &= \sqrt{\left[\left(\frac{m_c}{m_c + m_t}\right)v_c\right]^2 + \left[\left(\frac{m_t}{m_c + m_t}\right)v_t\right]^2} \\
&= \sqrt{\left[\left(\frac{1200\text{ kg}}{4200\text{ kg}}\right)\left(16.67\ \frac{\text{m}}{\text{s}}\right)\right]^2 + \left[\left(\frac{3000\text{ kg}}{4200\text{ kg}}\right)\left(11.1\ \frac{\text{m}}{\text{s}}\right)\right]^2} \\
&= \sqrt{\left(4.76\ \frac{\text{m}}{\text{s}}\right)^2 + \left(7.93\ \frac{\text{m}}{\text{s}}\right)^2} \\
&= 9.25\ \frac{\text{m}}{\text{s}} \approx 33.3\ \frac{\text{km}}{\text{hr}}.
\end{aligned}
$$

As for its direction, using the angle shown in the figure,

$$\theta = \tan^{-1}\left(\frac{v_{\text{w},\,x}}{v_{\text{w},\,y}}\right) = \tan^{-1}\left(\frac{7.93\text{ m/s}}{4.76\text{ m/s}}\right) = 59°.$$

This angle is east of north, or $31°$ counterclockwise from the +*x*-direction.

Significance

As a practical matter, accident investigators usually work in the "opposite direction"; they measure the distance of skid marks on the road (which gives the stopping distance) and use the work-energy theorem along with conservation of momentum to determine the speeds and directions of the cars prior to the collision. We saw that analysis in an earlier section.

9.9 Check Your Understanding Suppose the initial velocities were *not* at right angles to each other. How would this change both the physical result and the mathematical analysis of the collision?

Example 9.15

Exploding Scuba Tank

A common scuba tank is an aluminum cylinder that weighs 31.7 pounds empty (**Figure 9.25**). When full of compressed air, the internal pressure is between 2500 and 3000 psi (pounds per square inch). Suppose such a tank, which had been sitting motionless, suddenly explodes into three pieces. The first piece, weighing 10 pounds, shoots off horizontally at 235 miles per hour; the second piece (7 pounds) shoots off at 172 miles per hour, also in the horizontal plane, but at a $19°$ angle to the first piece. What is the mass and initial velocity of the third piece? (Do all work, and express your final answer, in SI units.)

Figure 9.25 A scuba tank explodes into three pieces.

Strategy

To use conservation of momentum, we need a closed system. If we define the system to be the scuba tank, this is not a closed system, since gravity is an external force. However, the problem asks for the just the initial velocity of the third piece, so we can neglect the effect of gravity and consider the tank by itself as a closed system. Notice that, for this system, the initial momentum vector is zero.

We choose a coordinate system where all the motion happens in the *xy*-plane. We then write down the equations for conservation of momentum in each direction, thus obtaining the *x*- and *y*-components of the momentum of the third piece, from which we obtain its magnitude (via the Pythagorean theorem) and its direction. Finally, dividing this momentum by the mass of the third piece gives us the velocity.

Solution

First, let's get all the conversions to SI units out of the way:

$$31.7\text{ lb} \times \frac{1\text{ kg}}{2.2\text{ lb}} \rightarrow 14.4\text{ kg}$$

$$10\text{ lb} \rightarrow 4.5\text{ kg}$$

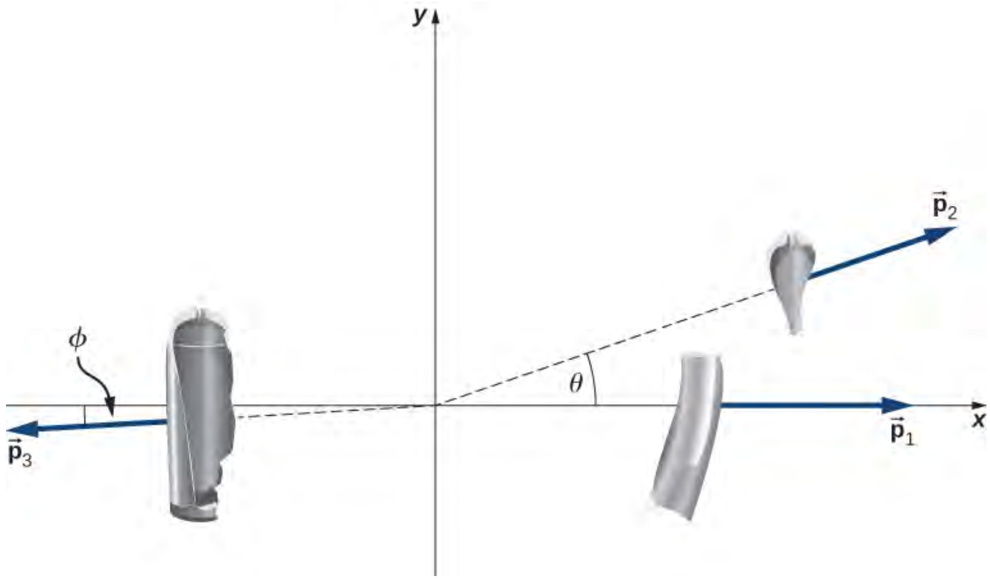
$$235\,\frac{\text{miles}}{\text{hour}} \times \frac{1\text{ hour}}{3600\text{ s}} \times \frac{1609\text{ m}}{\text{mile}} = 105\,\frac{\text{m}}{\text{s}}$$

$$7\text{ lb} \rightarrow 3.2\text{ kg}$$

$$172\,\frac{\text{mile}}{\text{hour}} = 77\,\frac{\text{m}}{\text{s}}$$

$$m_3 = 14.4\text{ kg} - (4.5\text{ kg} + 3.2\text{ kg}) = 6.7\text{ kg}.$$

Now apply conservation of momentum in each direction.



x-direction:

$$\begin{aligned} p_{f,x} &= p_{0,x} \\ p_{1,x} + p_{2,x} + p_{3,x} &= 0 \\ m_1 v_{1,x} + m_2 v_{2,x} + p_{3,x} &= 0 \\ p_{3,x} &= -m_1 v_{1,x} - m_2 v_{2,x} \end{aligned}$$

y-direction:

$$\begin{aligned} p_{f,y} &= p_{0,y} \\ p_{1,y} + p_{2,y} + p_{3,y} &= 0 \\ m_1 v_{1,y} + m_2 v_{2,y} + p_{3,y} &= 0 \\ p_{3,y} &= -m_1 v_{1,y} - m_2 v_{2,y} \end{aligned}$$

From our chosen coordinate system, we write the *x*-components as

$$\begin{aligned} p_{3,\,x} &= -m_1 v_1 - m_2 v_2 \cos\theta \\ &= -(14.5 \text{ kg})\left(105 \tfrac{\text{m}}{\text{s}}\right) - (4.5 \text{ kg})\left(77 \tfrac{\text{m}}{\text{s}}\right)\cos(19°) \\ &= -1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}. \end{aligned}$$

For the y-direction, we have

$$\begin{aligned} p_{3y} &= 0 - m_2 v_2 \sin\theta \\ &= -(4.5 \text{ kg})\left(77 \tfrac{\text{m}}{\text{s}}\right)\sin(19°) \\ &= -113 \frac{\text{kg} \cdot \text{m}}{\text{s}}. \end{aligned}$$

This gives the magnitude of p_3:

$$\begin{aligned} p_3 &= \sqrt{p_{3,x}^2 + p_{3,y}^2} \\ &= \sqrt{\left(-1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}\right)^2 + \left(-113 \frac{\text{kg} \cdot \text{m}}{\text{s}}\right)} \\ &= 1854 \frac{\text{kg} \cdot \text{m}}{\text{s}}. \end{aligned}$$

The velocity of the third piece is therefore

$$v_3 = \frac{p_3}{m_3} = \frac{1854 \frac{\text{kg} \cdot \text{m}}{\text{s}}}{6.7 \text{ kg}} = 277 \frac{\text{m}}{\text{s}}.$$

The direction of its velocity vector is the same as the direction of its momentum vector:

$$\phi = \tan^{-1}\left(\frac{p_{3,y}}{p_{3,x}}\right) = \tan^{-1}\left(\frac{113 \frac{\text{kg} \cdot \text{m}}{\text{s}}}{1850 \frac{\text{kg} \cdot \text{m}}{\text{s}}}\right) = 3.5°.$$

Because ϕ is below the $-x$-axis, the actual angle is $183.5°$ from the $+x$-direction.

Significance

The enormous velocities here are typical; an exploding tank of any compressed gas can easily punch through the wall of a house and cause significant injury, or death. Fortunately, such explosions are extremely rare, on a percentage basis.

9.10 Check Your Understanding Notice that the mass of the air in the tank was neglected in the analysis and solution. How would the solution method changed if the air was included? How large a difference do you think it would make in the final answer?

9.6 | Center of Mass

Learning Objectives

By the end of this section, you will be able to:

- Explain the meaning and usefulness of the concept of center of mass
- Calculate the center of mass of a given system
- Apply the center of mass concept in two and three dimensions
- Calculate the velocity and acceleration of the center of mass

We have been avoiding an important issue up to now: When we say that an object moves (more correctly, accelerates) in a way that obeys Newton's second law, we have been ignoring the fact that all objects are actually made of many constituent particles. A car has an engine, steering wheel, seats, passengers; a football is leather and rubber surrounding air; a brick is made of atoms. There are many different types of particles, and they are generally not distributed uniformly in the object. How do we include these facts into our calculations?

Then too, an extended object might change shape as it moves, such as a water balloon or a cat falling (Figure 9.26). This implies that the constituent particles are applying internal forces on each other, in addition to the external force that is acting on the object as a whole. We want to be able to handle this, as well.

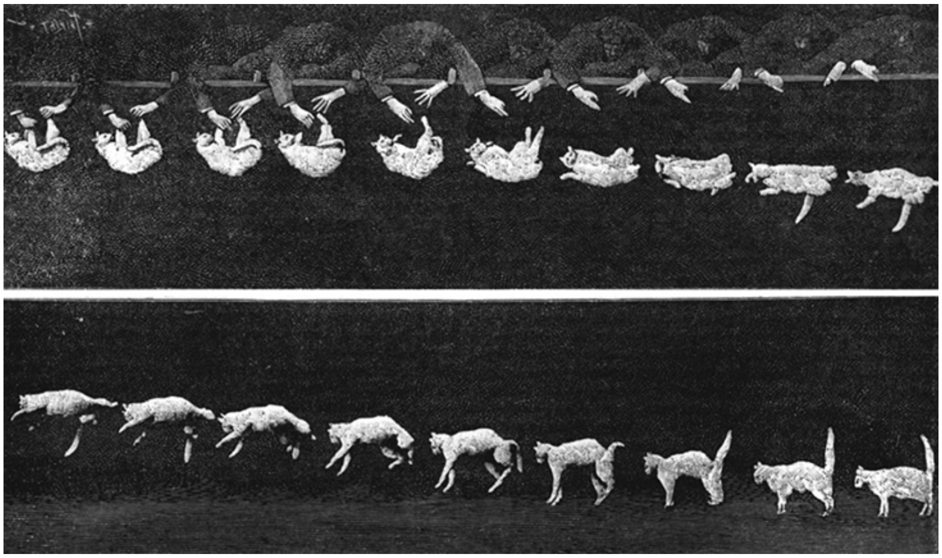


Figure 9.26 As the cat falls, its body performs complicated motions so it can land on its feet, but one point in the system moves with the simple uniform acceleration of gravity.

The problem before us, then, is to determine what part of an extended object is obeying Newton's second law when an external force is applied and to determine how the motion of the object as a whole is affected by both the internal and external forces.

Be warned: To treat this new situation correctly, we must be rigorous and completely general. We won't make any assumptions about the nature of the object, or of its constituent particles, or either the internal or external forces. Thus, the arguments will be complex.

Internal and External Forces

Suppose we have an extended object of mass M, made of N interacting particles. Let's label their masses as m_j, where $j = 1, 2, 3, \ldots, N$. Note that

$$M = \sum_{j=1}^{N} m_j. \tag{9.19}$$

If we apply some net **external force** $\vec{\mathbf{F}}_{\text{ext}}$ on the object, every particle experiences some "share" or some fraction of that external force. Let:

$$\vec{\mathbf{f}}_j^{\text{ext}} = \text{ the fraction of the external force that the } j\text{th particle experiences.}$$

Notice that these fractions of the total force are not necessarily equal; indeed, they virtually never are. (They *can* be, but they usually aren't.) In general, therefore,

$$\vec{\mathbf{f}}_1^{\text{ext}} \neq \vec{\mathbf{f}}_2^{\text{ext}} \neq \cdots \neq \vec{\mathbf{f}}_N^{\text{ext}}.$$

Next, we assume that each of the particles making up our object can interact (apply forces on) every other particle of the object. We won't try to guess what kind of forces they are; but since these forces are the result of particles of the object acting on other particles of the same object, we refer to them as **internal forces** $\vec{\mathbf{f}}_j^{\text{int}}$; thus:

$\vec{\mathbf{f}}_j^{\text{int}} =$ the net internal force that the *j*th particle experiences from all the other particles that make up the object.

Now, the *net* force, internal plus external, on the *j*th particle is the vector sum of these:

$$\vec{\mathbf{f}}_j = \vec{\mathbf{f}}_j^{\text{int}} + \vec{\mathbf{f}}_j^{\text{ext}}. \tag{9.20}$$

where again, this is for all N particles; $j = 1, 2, 3, \ldots, N$.

As a result of this fractional force, the momentum of each particle gets changed:

$$\begin{aligned} \vec{\mathbf{f}}_j &= \frac{d\vec{\mathbf{p}}_j}{dt} \\ \vec{\mathbf{f}}_j^{\text{int}} + \vec{\mathbf{f}}_j^{\text{ext}} &= \frac{d\vec{\mathbf{p}}_j}{dt}. \end{aligned} \tag{9.21}$$

The net force $\vec{\mathbf{F}}$ on the *object* is the vector sum of these forces:

$$\begin{aligned} \vec{\mathbf{F}}_{\text{net}} &= \sum_{j=1}^{N} \left(\vec{\mathbf{f}}_{\mathbf{j}}^{\textbf{int}} + \vec{\mathbf{f}}_{\mathbf{j}}^{\textbf{ext}} \right) \\ &= \sum_{j=1}^{N} \vec{\mathbf{f}}_{\mathbf{j}}^{\textbf{int}} + \sum_{j=1}^{N} \vec{\mathbf{f}}_{\mathbf{j}}^{\textbf{ext}}. \end{aligned} \tag{9.22}$$

This net force changes the momentum of the object as a whole, and the net change of momentum of the object must be the vector sum of all the individual changes of momentum of all of the particles:

$$\vec{\mathbf{F}}_{\text{net}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt}. \tag{9.23}$$

Combining **Equation 9.22** and **Equation 9.23** gives

$$\sum_{j=1}^{N} \vec{\mathbf{f}}_j^{\text{int}} + \sum_{j=1}^{N} \vec{\mathbf{f}}_j^{\text{ext}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt}. \tag{9.24}$$

Let's now think about these summations. First consider the internal forces term; remember that each $\vec{\mathbf{f}}_j^{\text{int}}$ is the force on the *j*th particle from the other particles in the object. But by Newton's third law, for every one of these forces, there must be another force that has the same magnitude, but the opposite sign (points in the opposite direction). These forces do not cancel; however, that's not what we're doing in the summation. Rather, we're simply *mathematically adding up* all the internal force vectors. That is, in general, the internal forces for any individual part of the object won't cancel, but when all the internal forces are added up, the internal forces must cancel in pairs. It follows, therefore, that the sum of all the internal forces must be zero:

$$\sum_{j=1}^{N} \vec{\mathbf{f}}_j^{\text{int}} = 0.$$

(This argument is subtle, but crucial; take plenty of time to completely understand it.)

For the external forces, this summation is simply the total external force that was applied to the whole object:

$$\sum_{j=1}^{N} \vec{\mathbf{f}}_j^{\text{ext}} = \vec{\mathbf{F}}_{\text{ext}}.$$

As a result,

$$\vec{\mathbf{F}}_{\text{ext}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt}. \tag{9.25}$$

This is an important result. **Equation 9.25** tells us that the total change of momentum of the entire object (all N particles) is due only to the external forces; the internal forces do not change the momentum of the object as a whole. This is why you can't lift yourself in the air by standing in a basket and pulling up on the handles: For the system of you + basket, your upward pulling force is an internal force.

Force and Momentum

Remember that our actual goal is to determine the equation of motion for the entire object (the entire system of particles). To that end, let's define:

$\vec{\mathbf{p}}_{\text{CM}} =$ the total momentum of the system of N particles (the reason for the subscript will become clear shortly)

Then we have

$$\vec{\mathbf{p}}_{\text{CM}} \equiv \sum_{j=1}^{N} \vec{\mathbf{p}}_j,$$

and therefore **Equation 9.25** can be written simply as

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}_{\text{CM}}}{dt}. \tag{9.26}$$

Since this change of momentum is caused by only the net external force, we have dropped the "ext" subscript.

This is Newton's second law, but now for the entire extended object. If this feels a bit anticlimactic, remember what is hiding inside it: $\vec{\mathbf{p}}_{\text{CM}}$ is the vector sum of the momentum of (in principle) hundreds of thousands of billions of billions of particles (6.02×10^{23}), all caused by one simple net external force—a force that you can calculate.

Center of Mass

Our next task is to determine what part of the extended object, if any, is obeying **Equation 9.26**.

It's tempting to take the next step; does the following equation mean anything?

$$\vec{\mathbf{F}} = M\vec{\mathbf{a}} \tag{9.27}$$

If it *does* mean something (acceleration of what, exactly?), then we could write

$$M\vec{\mathbf{a}} = \frac{d\vec{\mathbf{p}}_{\text{CM}}}{dt}$$

and thus

$$M\vec{\mathbf{a}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt} = \frac{d}{dt}\sum_{j=1}^{N} \vec{\mathbf{p}}_j.$$

which follows because the derivative of a sum is equal to the sum of the derivatives.

Now, $\vec{\mathbf{p}}_j$ is the momentum of the jth particle. Defining the positions of the constituent particles (relative to some coordinate system) as $\vec{\mathbf{r}}_{\mathbf{j}} = \left(x_j, y_j, z_j\right)$, we thus have

$$\vec{\mathbf{p}}_j = m_j \vec{\mathbf{v}}_j = m_j \frac{d\vec{\mathbf{r}}_j}{dt}.$$

Substituting back, we obtain

$$\begin{aligned} M \vec{\mathbf{a}} &= \frac{d}{dt} \sum_{j=1}^{N} m_j \frac{d \vec{\mathbf{r}}_j}{dt} \\ &= \frac{d^2}{dt^2} \sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j. \end{aligned}$$

Dividing both sides by M (the total mass of the extended object) gives us

$$\vec{\mathbf{a}} = \frac{d^2}{dt^2}\left(\frac{1}{M} \sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j\right). \tag{9.28}$$

Thus, the point in the object that traces out the trajectory dictated by the applied force in **Equation 9.27** is inside the parentheses in **Equation 9.28**.

Looking at this calculation, notice that (inside the parentheses) we are calculating the product of each particle's mass with its position, adding all N of these up, and dividing this sum by the total mass of particles we summed. This is reminiscent of an average; inspired by this, we'll (loosely) interpret it to be the weighted average position of the mass of the extended object. It's actually called the **center of mass** of the object. Notice that the position of the center of mass has units of meters; that suggests a definition:

$$\vec{\mathbf{r}}_{\text{CM}} \equiv \frac{1}{M} \sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j. \tag{9.29}$$

So, the point that obeys **Equation 9.26** (and therefore **Equation 9.27** as well) is the center of mass of the object, which is located at the position vector $\vec{\mathbf{r}}_{\text{CM}}$.

It may surprise you to learn that there does not have to be any actual mass at the center of mass of an object. For example, a hollow steel sphere with a vacuum inside it is spherically symmetrical (meaning its mass is uniformly distributed about the center of the sphere); all of the sphere's mass is out on its surface, with no mass inside. But it can be shown that the center of mass of the sphere is at its geometric center, which seems reasonable. Thus, there is no mass at the position of the center of mass of the sphere. (Another example is a doughnut.) The procedure to find the center of mass is illustrated in **Figure 9.27**.

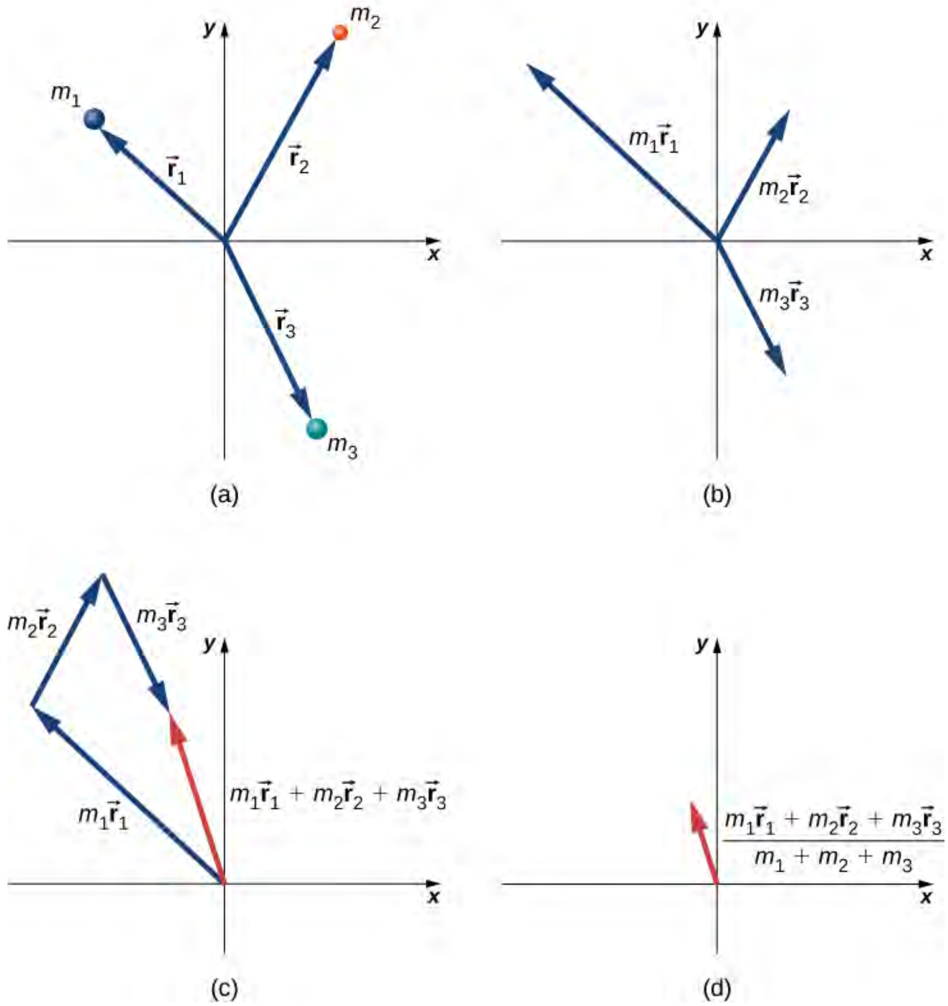


Figure 9.27 Finding the center of mass of a system of three different particles. (a) Position vectors are created for each object. (b) The position vectors are multiplied by the mass of the corresponding object. (c) The scaled vectors from part (b) are added together. (d) The final vector is divided by the total mass. This vector points to the center of mass of the system. Note that no mass is actually present at the center of mass of this system.

Since $\vec{\mathbf{r}}_j = x_j \hat{\mathbf{i}} + y_j \hat{\mathbf{j}} + z_j \hat{\mathbf{k}}$, it follows that:

$$r_{\text{CM},x} = \frac{1}{M} \sum_{j=1}^{N} m_j x_j \tag{9.30}$$

$$r_{\text{CM},y} = \frac{1}{M} \sum_{j=1}^{N} m_j y_j \tag{9.31}$$

$$r_{\text{CM},z} = \frac{1}{M} \sum_{j=1}^{N} m_j z_j \tag{9.32}$$

and thus

$$\vec{\mathbf{r}}_{\text{CM}} = r_{\text{CM},x} \hat{\mathbf{i}} + r_{\text{CM},y} \hat{\mathbf{j}} + r_{\text{CM},z} \hat{\mathbf{k}}$$
$$r_{\text{CM}} = \left| \vec{\mathbf{r}}_{\text{CM}} \right| = \left(r_{\text{CM},x}^2 + r_{\text{CM},y}^2 + r_{\text{CM},z}^2 \right)^{1/2}.$$

Therefore, you can calculate the components of the center of mass vector individually.

Finally, to complete the kinematics, the instantaneous velocity of the center of mass is calculated exactly as you might suspect:

$$\vec{\mathbf{v}}_{\mathrm{CM}} = \frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{v}}_j \tag{9.33}$$

and this, like the position, has *x*-, *y*-, and *z*-components.

To calculate the center of mass in actual situations, we recommend the following procedure:

Problem-Solving Strategy: Calculating the Center of Mass

The center of mass of an object is a position vector. Thus, to calculate it, do these steps:

1. Define your coordinate system. Typically, the origin is placed at the location of one of the particles. This is not required, however.
2. Determine the *x*, *y*, *z*-coordinates of each particle that makes up the object.
3. Determine the mass of each particle, and sum them to obtain the total mass of the object. Note that the mass of the object at the origin *must* be included in the total mass.
4. Calculate the *x*-, *y*-, and *z*-components of the center of mass vector, using **Equation 9.30**, **Equation 9.31**, and **Equation 9.32**.
5. If required, use the Pythagorean theorem to determine its magnitude.

Here are two examples that will give you a feel for what the center of mass is.

Example 9.16

Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use **Equation 9.29** with just $N = 2$ objects. We use a subscript "e" to refer to Earth, and subscript "m" to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, **Equation 9.29** becomes

$$R = \frac{m_{\mathrm{e}} r_{\mathrm{e}} + m_{\mathrm{m}} r_{\mathrm{m}}}{m_{\mathrm{e}} + m_{\mathrm{m}}}.$$

From **Appendix D**,

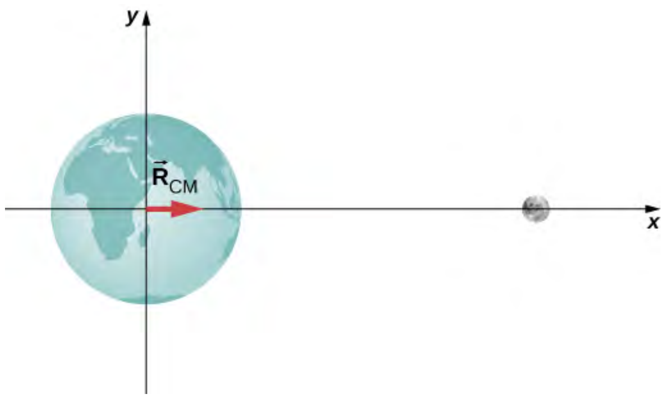
$$m_{\mathrm{e}} = 5.97 \times 10^{24}\ \mathrm{kg}$$
$$m_{\mathrm{m}} = 7.36 \times 10^{22}\ \mathrm{kg}$$
$$r_{\mathrm{m}} = 3.82 \times 10^{5}\ \mathrm{m}.$$

We defined the center of Earth as the origin, so $r_{\mathrm{e}} = 0\ \mathrm{m}$. Inserting these into the equation for *R* gives

$$\begin{aligned} R &= \frac{\left(5.97 \times 10^{24} \text{ kg}\right)(0 \text{ m}) + \left(7.36 \times 10^{22} \text{ kg}\right)\left(3.82 \times 10^{8} \text{ m}\right)}{5.98 \times 10^{24} \text{ kg} + 7.36 \times 10^{22} \text{ kg}} \\ &= 4.64 \times 10^{6} \text{ m}. \end{aligned}$$

Significance

The radius of Earth is 6.37×10^6 m, so the center of mass of the Earth-moon system is (6.37 − 4.64) $\times 10^6 \text{ m} = 1.73 \times 10^6 \text{ m} = 1730 \text{ km}$ (roughly 1080 miles) *below* the surface of Earth. The location of the center of mass is shown (not to scale).



9.11 Check Your Understanding Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

Example 9.17

Center of Mass of a Salt Crystal

Figure 9.28 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the *unit cell*) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36×10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates $\left(r_{\text{CM},x}, r_{\text{CM},y}, r_{\text{CM},z}\right)$, or by r_{CM} and two angles.

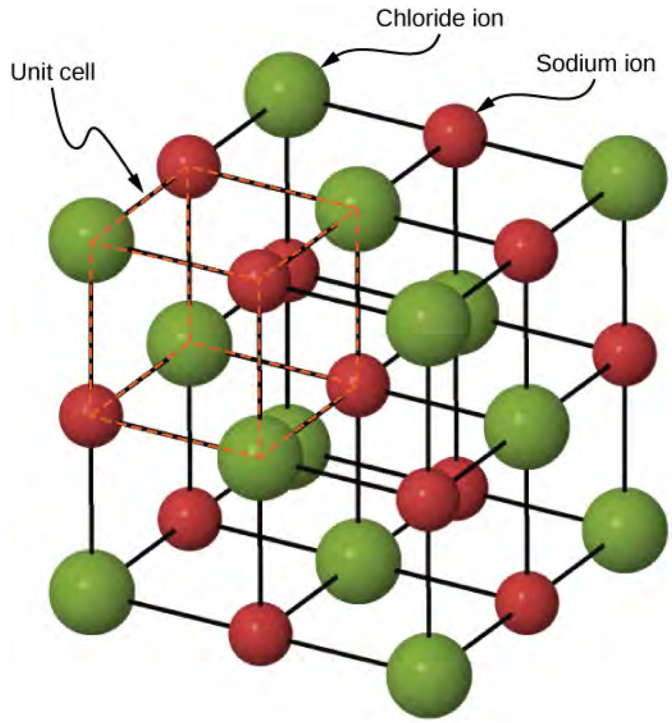


Figure 9.28 A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply **Equation 9.30**, **Equation 9.31**, and **Equation 9.32** (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. **Figure 9.29** shows the coordinate system.

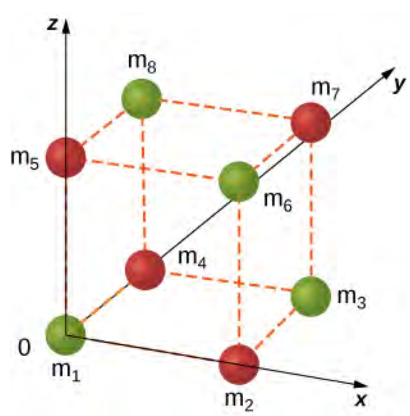


Figure 9.29 A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so $N = 8$:

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\sum_{j=1}^{8} m_j \vec{\mathbf{r}}_j.$$

The mass of each of the chloride ions is

$$35.453\text{u} \times \frac{1.660 \times 10^{-27}\ \text{kg}}{\text{u}} = 5.885 \times 10^{-26}\ \text{kg}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26}\ \text{kg}.$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26}\ \text{kg}.$$

The total mass of the unit cell is therefore

$$M = (4)\left(5.885 \times 10^{-26}\ \text{kg}\right) + (4)\left(3.816 \times 10^{-26}\ \text{kg}\right) = 3.880 \times 10^{-25}\ \text{kg}.$$

From the geometry, the locations are

$$\begin{aligned}
\vec{\mathbf{r}}_1 &= 0 \\
\vec{\mathbf{r}}_2 &= \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{i}} \\
\vec{\mathbf{r}}_3 &= r_{3x}\,\hat{\mathbf{i}} + r_{3y}\,\hat{\mathbf{j}} = \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{i}} + \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{j}} \\
\vec{\mathbf{r}}_4 &= \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{j}} \\
\vec{\mathbf{r}}_5 &= \left(2.36 \times 10^{-10}\ \text{m}\right)\vec{\mathbf{k}} \\
\vec{\mathbf{r}}_6 &= r_{6x}\,\hat{\mathbf{i}} + r_{6z}\,\hat{\mathbf{k}} = \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{i}} + \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{k}} \\
\vec{\mathbf{r}}_7 &= r_{7x}\,\hat{\mathbf{i}} + r_{7y}\,\hat{\mathbf{j}} + r_{7z}\,\hat{\mathbf{k}} = \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{i}} + \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{j}} + \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{k}} \\
\vec{\mathbf{r}}_8 &= r_{8y}\,\hat{\mathbf{j}} + r_{8z}\,\hat{\mathbf{k}} = \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{j}} + \left(2.36 \times 10^{-10}\ \text{m}\right)\hat{\mathbf{k}}.
\end{aligned}$$

Substituting:

$$\begin{aligned}
\left|\vec{\mathbf{r}}_{\text{CM},x}\right| &= \sqrt{r_{\text{CM},x}^2 + r_{\text{CM},y}^2 + r_{\text{CM},z}^2} \\
&= \frac{1}{M}\sum_{j=1}^{8} m_j (r_x)_j \\
&= \frac{1}{M}(m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\
&= \frac{1}{3.8804 \times 10^{-25}\ \text{kg}}\Big[\left(5.885 \times 10^{-26}\ \text{kg}\right)(0\ \text{m}) + \left(3.816 \times 10^{-26}\ \text{kg}\right)\left(2.36 \times 10^{-10}\ \text{m}\right) \\
&\quad + \left(5.885 \times 10^{-26}\ \text{kg}\right)\left(2.36 \times 10^{-10}\ \text{m}\right) \\
&\quad + \left(3.816 \times 10^{-26}\ \text{kg}\right)\left(2.36 \times 10^{-10}\ \text{m}\right) + 0 + 0 \\
&\quad + \left(3.816 \times 10^{-26}\ \text{kg}\right)\left(2.36 \times 10^{-10}\ \text{m}\right) + 0\Big] \\
&= 1.18 \times 10^{-10}\ \text{m}.
\end{aligned}$$

Similar calculations give $r_{\text{CM},y} = r_{\text{CM},z} = 1.18 \times 10^{-10}\ \text{m}$ (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned} r_{\text{CM}} &= \sqrt{r_{\text{CM},x}^2 + r_{\text{CM},y}^2 + r_{\text{CM},z}^2} \\ &= \left(1.18 \times 10^{-10}\text{ m}\right)\sqrt{3} \\ &= 2.044 \times 10^{-10}\text{ m}. \end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{\mathbf{r}}_{\text{CM}}$. Let ϕ be the angle in the *x*,*y*-plane, measured from the +*x*-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{\text{CM},y}}{r_{\text{CM},x}}\right) = 45°.$$

Let θ be the angle in the *y*,*z*-plane, measured downward from the +*z*-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45°.$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry.

9.12 Check Your Understanding Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a *huge* number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in **Equation 9.29** to be zero. However, you must still include the mass of the object at your origin in your calculation of *M*, the total mass **Equation 9.19**. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

Center of Mass of Continuous Objects

If the object in question has its mass distributed uniformly in space, rather than as a collection of discrete particles, then $m_j \to dm$, and the summation becomes an integral:

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int \vec{\mathbf{r}}\, dm. \quad (9.34)$$

In this context, *r* is a characteristic dimension of the object (the radius of a sphere, the length of a long rod). To generate an integrand that can actually be calculated, you need to express the differential mass element *dm* as a function of the mass density of the continuous object, and the dimension *r*. An example will clarify this.

Example 9.18

CM of a Uniform Thin Hoop

Find the center of mass of a uniform thin hoop (or ring) of mass M and radius r.

Strategy

First, the hoop's symmetry suggests the center of mass should be at its geometric center. If we define our coordinate system such that the origin is located at the center of the hoop, the integral should evaluate to zero.

We replace dm with an expression involving the density of the hoop and the radius of the hoop. We then have an expression we can actually integrate. Since the hoop is described as "thin," we treat it as a one-dimensional object, neglecting the thickness of the hoop. Therefore, its density is expressed as the number of kilograms of material per meter. Such a density is called a **linear mass density**, and is given the symbol λ; this is the Greek letter "lambda," which is the equivalent of the English letter "l" (for "linear").

Since the hoop is described as uniform, this means that the linear mass density λ is constant. Thus, to get our expression for the differential mass element dm, we multiply λ by a differential length of the hoop, substitute, and integrate (with appropriate limits for the definite integral).

Solution

First, define our coordinate system and the relevant variables (**Figure 9.30**).

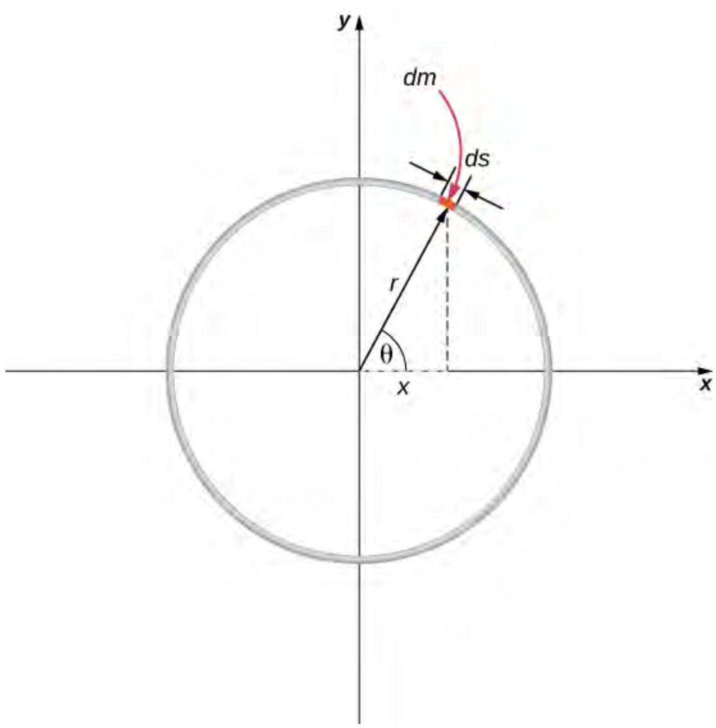


Figure 9.30 Finding the center of mass of a uniform hoop. We express the coordinates of a differential piece of the hoop, and then integrate around the hoop.

The center of mass is calculated with **Equation 9.34**:

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int_a^b \vec{\mathbf{r}}\, dm.$$

We have to determine the limits of integration a and b. Expressing $\vec{\mathbf{r}}$ in component form gives us

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int_a^b \left[(r\cos\theta)\,\hat{\mathbf{i}} + (r\sin\theta)\,\hat{\mathbf{j}} \right] dm.$$

In the diagram, we highlighted a piece of the hoop that is of differential length *ds*; it therefore has a differential mass $dm = \lambda ds$. Substituting:

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int_a^b \left[(r\cos\theta)\,\hat{\mathbf{i}} + (r\sin\theta)\,\hat{\mathbf{j}} \right] \lambda ds.$$

However, the arc length *ds* subtends a differential angle $d\theta$, so we have

$$ds = r d\theta$$

and thus

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\int_a^b \left[(r\cos\theta)\,\hat{\mathbf{i}} + (r\sin\theta)\,\hat{\mathbf{j}} \right] \lambda r d\theta.$$

One more step: Since λ is the linear mass density, it is computed by dividing the total mass by the length of the hoop:

$$\lambda = \frac{M}{2\pi r}$$

giving us

$$\begin{aligned} \vec{\mathbf{r}}_{\text{CM}} &= \frac{1}{M}\int_a^b \left[(r\cos\theta)\,\hat{\mathbf{i}} + (r\sin\theta)\,\hat{\mathbf{j}} \right] \left(\frac{M}{2\pi r}\right) r d\theta \\ &= \frac{1}{2\pi}\int_a^b \left[(r\cos\theta)\,\hat{\mathbf{i}} + (r\sin\theta)\,\hat{\mathbf{j}} \right] d\theta. \end{aligned}$$

Notice that the variable of integration is now the angle θ. This tells us that the limits of integration (around the circular hoop) are $\theta = 0$ to $\theta = 2\pi$, so $a = 0$ and $b = 2\pi$. Also, for convenience, we separate the integral into the *x*- and *y*-components of $\vec{\mathbf{r}}_{\text{CM}}$. The final integral expression is

$$\begin{aligned} \vec{\mathbf{r}}_{\text{CM}} &= r_{\text{CM},x}\,\hat{\mathbf{i}} + r_{\text{CM},y}\,\hat{\mathbf{j}} \\ &= \left[\frac{1}{2\pi}\int_0^{2\pi} (r\cos\theta) d\theta \right] \hat{\mathbf{i}} + \left[\frac{1}{2\pi}\int_0^{2\pi} (r\sin\theta) d\theta \right] \hat{\mathbf{j}} \\ &= 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} = \vec{\mathbf{0}} \end{aligned}$$

as expected.

Center of Mass and Conservation of Momentum

How does all this connect to conservation of momentum?

Suppose you have *N* objects with masses $m_1, m_2, m_3, \ldots m_N$ and initial velocities $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \ldots, \vec{\mathbf{v}}_N$. The center of mass of the objects is

$$\vec{\mathbf{r}}_{\text{CM}} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j.$$

Its velocity is

$$\vec{\mathbf{v}}_{\text{CM}} = \frac{d\,\vec{\mathbf{r}}_{\text{CM}}}{dt} = \frac{1}{M}\sum_{j=1}^{N} m_j \frac{d\,\vec{\mathbf{r}}_j}{dt} \tag{9.35}$$

and thus the initial momentum of the center of mass is

$$\begin{aligned}\left[M\frac{d\,\vec{\mathbf{r}}_{\text{CM}}}{dt}\right]_{\text{i}} &= \sum_{j=1}^{N} m_j \frac{d\,\vec{\mathbf{r}}_{j,\,\text{i}}}{dt} \\ M\,\vec{\mathbf{v}}_{\text{CM,i}} &= \sum_{j=1}^{N} m_j\,\vec{\mathbf{v}}_{j,\,\text{i}}.\end{aligned}$$

After these masses move and interact with each other, the momentum of the center of mass is

$$M\,\vec{\mathbf{v}}_{\text{CM,f}} = \sum_{j=1}^{N} m_j\,\vec{\mathbf{v}}_{j,\,\text{f}}.$$

But conservation of momentum tells us that the right-hand side of both equations must be equal, which says

$$M\,\vec{\mathbf{v}}_{\text{CM,f}} = M\,\vec{\mathbf{v}}_{\text{CM,i}}. \tag{9.36}$$

This result implies that conservation of momentum is expressed in terms of the center of mass of the system. Notice that as an object moves through space with no net external force acting on it, an individual particle of the object may accelerate in various directions, with various magnitudes, depending on the net internal force acting on that object at any time. (Remember, it is only the vector sum of all the internal forces that vanishes, not the internal force on a single particle.) Thus, such a particle's momentum will not be constant—but the momentum of the entire extended object will be, in accord with **Equation 9.36**.

Equation 9.36 implies another important result: Since M represents the mass of the entire system of particles, it is necessarily constant. (If it isn't, we don't have a closed system, so we can't expect the system's momentum to be conserved.) As a result, **Equation 9.36** implies that, for a closed system,

$$\vec{\mathbf{v}}_{\text{CM,f}} = \vec{\mathbf{v}}_{\text{CM,i}}. \tag{9.37}$$

That is to say, *in the absence of an external force, the velocity of the center of mass never changes*.

You might be tempted to shrug and say, "Well yes, that's just Newton's first law," but remember that Newton's first law discusses the constant velocity of a particle, whereas **Equation 9.37** applies to the center of mass of a (possibly vast) collection of interacting particles, and that there may not be any particle at the center of mass at all! So, this really is a remarkable result.

Example 9.19

Fireworks Display

When a fireworks rocket explodes, thousands of glowing fragments fly outward in all directions, and fall to Earth in an elegant and beautiful display (**Figure 9.31**). Describe what happens, in terms of conservation of momentum and center of mass.

Figure 9.31 These exploding fireworks are a vivid example of conservation of momentum and the motion of the center of mass.

The picture shows radial symmetry about the central points of the explosions; this suggests the idea of center of mass. We can also see the parabolic motion of the glowing particles; this brings to mind projectile motion ideas.

Solution

Initially, the fireworks rocket is launched and flies more or less straight upward; this is the cause of the more-or-less-straight, white trail going high into the sky below the explosion in the upper-right of the picture (the yellow explosion). This trail is not parabolic because the explosive shell, during its launch phase, is actually a rocket; the impulse applied to it by the ejection of the burning fuel applies a force on the shell during the rise-time interval. (This is a phenomenon we will study in the next section.) The shell has multiple forces on it; thus, it is not in free-fall prior to the explosion.

At the instant of the explosion, the thousands of glowing fragments fly outward in a radially symmetrical pattern. The symmetry of the explosion is the result of all the internal forces summing to zero $\left(\sum_j \vec{\mathbf{f}}_j^{\text{int}} = 0\right)$; for every internal force, there is another that is equal in magnitude and opposite in direction.

However, as we learned above, these internal forces cannot change the momentum of the center of mass of the (now exploded) shell. Since the rocket force has now vanished, the center of mass of the shell is now a projectile (the only force on it is gravity), so its trajectory does become parabolic. The two red explosions on the left show the path of their centers of mass at a slightly longer time after explosion compared to the yellow explosion on the upper right.

In fact, if you look carefully at all three explosions, you can see that the glowing trails are not truly radially symmetric; rather, they are somewhat denser on one side than the other. Specifically, the yellow explosion and the lower middle explosion are slightly denser on their right sides, and the upper-left explosion is denser on its left side. This is because of the momentum of their centers of mass; the differing trail densities are due to the momentum each piece of the shell had at the moment of its explosion. The fragment for the explosion on the upper left of the picture had a momentum that pointed upward and to the left; the middle fragment's momentum pointed upward and slightly to the right; and the right-side explosion clearly upward and to the right (as evidenced by the white rocket exhaust trail visible below the yellow explosion).

Finally, each fragment is a projectile on its own, thus tracing out thousands of glowing parabolas.

Significance

In the discussion above, we said, "…the center of mass of the shell is now a projectile (the only force on it is gravity)…." This is not quite accurate, for there may not be any mass at all at the center of mass; in which case, there could not be a force acting on it. This is actually just verbal shorthand for describing the fact that the gravitational forces on all the particles act so that the center of mass changes position exactly as if all the mass of the shell were always located at the position of the center of mass.

9.13 Check Your Understanding How would the firework display change in deep space, far away from any source of gravity?

You may sometimes hear someone describe an explosion by saying something like, "the fragments of the exploded object always move in a way that makes sure that the center of mass continues to move on its original trajectory." This makes it sound as if the process is somewhat magical: how can it be that, in *every* explosion, it *always* works out that the fragments move in just the right way so that the center of mass' motion is unchanged? Phrased this way, it would be hard to believe no explosion ever does anything differently.

The explanation of this apparently astonishing coincidence is: We defined the center of mass precisely so this is exactly what we would get. Recall that first we defined the momentum of the system:

$$\vec{\mathbf{p}}_{\text{CM}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt}.$$

We then concluded that the net external force on the system (if any) changed this momentum:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}_{\text{CM}}}{dt}$$

and then—and here's the point—we defined an acceleration that would obey Newton's second law. That is, we demanded that we should be able to write

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{F}}}{M}$$

which requires that

$$\vec{\mathbf{a}} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j\right).$$

where the quantity inside the parentheses is the center of mass of our system. So, it's not astonishing that the center of mass obeys Newton's second law; we defined it so that it would.

9.7 | Rocket Propulsion

Learning Objectives

By the end of this section, you will be able to:

- Describe the application of conservation of momentum when the mass changes with time, as well as the velocity
- Calculate the speed of a rocket in empty space, at some time, given initial conditions
- Calculate the speed of a rocket in Earth's gravity field, at some time, given initial conditions

Now we deal with the case where the mass of an object is changing. We analyze the motion of a rocket, which changes its velocity (and hence its momentum) by ejecting burned fuel gases, thus causing it to accelerate in the opposite direction of the velocity of the ejected fuel (see **Figure 9.32**). Specifically: A fully fueled rocket ship in deep space has a total mass m_0 (this mass includes the initial mass of the fuel). At some moment in time, the rocket has a velocity $\vec{\mathbf{v}}$ and mass m;

this mass is a combination of the mass of the empty rocket and the mass of the remaining unburned fuel it contains. (We refer to *m* as the "instantaneous mass" and $\vec{v}$ as the "instantaneous velocity.") The rocket accelerates by burning the fuel it carries and ejecting the burned exhaust gases. If the burn rate of the fuel is constant, and the velocity at which the exhaust is ejected is also constant, what is the change of velocity of the rocket as a result of burning all of its fuel?

Figure 9.32 The space shuttle had a number of reusable parts. Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space shuttle was a complex assemblage of technologies, employing both solid and liquid fuel, and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to single-use rockets. (credit: modification of work by NASA)

Physical Analysis

Here's a description of what happens, so that you get a feel for the physics involved.

- As the rocket engines operate, they are continuously ejecting burned fuel gases, which have both mass and velocity, and therefore some momentum. By conservation of momentum, the rocket's momentum changes by this same amount (with the opposite sign). We will assume the burned fuel is being ejected at a constant rate, which means the rate of change of the rocket's momentum is also constant. By **Equation 9.9**, this represents a constant force on the rocket.
- However, as time goes on, the mass of the rocket (which includes the mass of the remaining fuel) continuously decreases. Thus, even though the force on the rocket is constant, the resulting acceleration is not; it is continuously increasing.
- So, the total change of the rocket's velocity will depend on the amount of mass of fuel that is burned, and that dependence is not linear.

The problem has the mass and velocity of the rocket changing; also, the total mass of ejected gases is changing. If we define our system to be the rocket + fuel, then this is a closed system (since the rocket is in deep space, there are no external forces acting on this system); as a result, momentum is conserved for this system. Thus, we can apply conservation of momentum to answer the question (**Figure 9.33**).

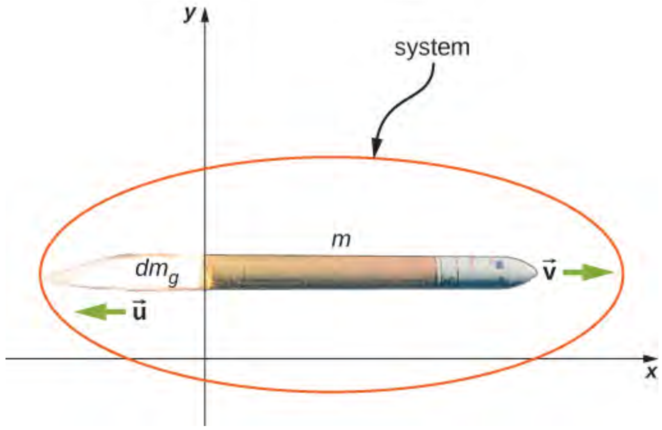


Figure 9.33 The rocket accelerates to the right due to the expulsion of some of its fuel mass to the left. Conservation of momentum enables us to determine the resulting change of velocity. The mass *m* is the instantaneous total mass of the rocket (i.e., mass of rocket body plus mass of fuel at that point in time). (credit: modification of work by NASA/Bill Ingalls)

At the same moment that the total instantaneous rocket mass is *m* (i.e., *m* is the mass of the rocket body plus the mass of the fuel at that point in time), we define the rocket's instantaneous velocity to be $\vec{\mathbf{v}} = v\,\hat{\mathbf{i}}$ (in the +*x*-direction); this velocity is measured relative to an inertial reference system (the Earth, for example). Thus, the initial momentum of the system is

$\vec{\mathbf{p}}_{\mathrm{i}} = mv\,\hat{\mathbf{i}}$.

The rocket's engines are burning fuel at a constant rate and ejecting the exhaust gases in the −*x*-direction. During an infinitesimal time interval *dt*, the engines eject a (positive) infinitesimal mass of gas dm_g at velocity $\vec{\mathbf{u}} = -u\,\hat{\mathbf{i}}$; note that although the rocket velocity $v\,\hat{\mathbf{i}}$ is measured with respect to Earth, the exhaust gas velocity is measured with respect to the (moving) rocket. Measured with respect to the Earth, therefore, the exhaust gas has velocity $(v-u)\,\hat{\mathbf{i}}$.

As a consequence of the ejection of the fuel gas, the rocket's mass decreases by dm_g, and its velocity increases by $dv\,\hat{\mathbf{i}}$. Therefore, including both the change for the rocket and the change for the exhaust gas, the final momentum of the system is

$$\begin{aligned}\vec{\mathbf{p}}_{\mathrm{f}} &= \vec{\mathbf{p}}_{\text{rocket}} + \vec{\mathbf{p}}_{\text{gas}} \\ &= \left(m - dm_g\right)(v + dv)\,\hat{\mathbf{i}} + dm_g(v-u)\,\hat{\mathbf{i}}.\end{aligned}$$

Since all vectors are in the *x*-direction, we drop the vector notation. Applying conservation of momentum, we obtain

$$\begin{aligned}&p_{\mathrm{i}} = p_{\mathrm{f}} \\ &mv = \left(m - dm_g\right)(v + dv) + dm_g(v - u) \\ &mv = mv + mdv - dm_g\,v - dm_g\,dv + dm_g\,v - dm_g\,u \\ &mdv = dm_g\,dv + dm_g\,v.\end{aligned}$$

Now, dm_g and *dv* are each very small; thus, their product $dm_g\,dv$ is very, very small, much smaller than the other two terms in this expression. We neglect this term, therefore, and obtain:

$$mdv = dm_g\,u.$$

Our next step is to remember that, since dm_g represents an increase in the mass of ejected gases, it must also represent a decrease of mass of the rocket:

$$dm_g = -dm.$$

Replacing this, we have

$$mdv = -dmu$$

or

$$dv = -u\frac{dm}{m}.$$

Integrating from the initial mass m_i to the final mass m of the rocket gives us the result we are after:

$$\begin{aligned} \int_{v_i}^{v} dv &= -u\int_{m_i}^{m} \frac{1}{m} dm \\ v - v_i &= u \ln\left(\frac{m_i}{m}\right) \end{aligned}$$

and thus our final answer is

$$\Delta v = u \ln\left(\frac{m_i}{m}\right). \tag{9.38}$$

This result is called the **rocket equation**. It was originally derived by the Soviet physicist Konstantin Tsiolkovsky in 1897. It gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass from m_0 down to m. As expected, the relationship between Δv and the change of mass of the rocket is nonlinear.

Problem-Solving Strategy: Rocket Propulsion

In rocket problems, the most common questions are finding the change of velocity due to burning some amount of fuel for some amount of time; or to determine the acceleration that results from burning fuel.

1. To determine the change of velocity, use the rocket equation **Equation 9.38**.
2. To determine the acceleration, determine the force by using the impulse-momentum theorem, using the rocket equation to determine the change of velocity.

Example 9.20

Thrust on a Spacecraft

A spacecraft is moving in gravity-free space along a straight path when its pilot decides to accelerate forward. He turns on the thrusters, and burned fuel is ejected at a constant rate of 2.0×10^2 kg/s, at a speed (relative to the rocket) of 2.5×10^2 m/s. The initial mass of the spacecraft and its unburned fuel is 2.0×10^4 kg, and the thrusters are on for 30 s.

a. What is the thrust (the force applied to the rocket by the ejected fuel) on the spacecraft?
b. What is the spacecraft's acceleration as a function of time?
c. What are the spacecraft's accelerations at t = 0, 15, 30, and 35 s?

Strategy

a. The force on the spacecraft is equal to the rate of change of the momentum of the fuel.
b. Knowing the force from part (a), we can use Newton's second law to calculate the consequent acceleration. The key here is that, although the force applied to the spacecraft is constant (the fuel is being ejected at a constant rate), the mass of the spacecraft isn't; thus, the acceleration caused by the force won't be constant. We expect to get a function $a(t)$, therefore.
c. We'll use the function we obtain in part (b), and just substitute the numbers given. Important: We expect that the acceleration will get larger as time goes on, since the mass being accelerated is continuously

decreasing (fuel is being ejected from the rocket).

Solution

a. The momentum of the ejected fuel gas is

$$p = m_g v.$$

The ejection velocity $v = 2.5 \times 10^2\,\text{m/s}$ is constant, and therefore the force is

$$F = \frac{dp}{dt} = v\frac{dm_g}{dt} = -v\frac{dm}{dt}.$$

Now, $\frac{dm_g}{dt}$ is the rate of change of the mass of the fuel; the problem states that this is $2.0 \times 10^2\,\text{kg/s}$. Substituting, we get

$$\begin{aligned} F &= v\frac{dm_g}{dt} \\ &= \left(2.5 \times 10^2\ \frac{\text{m}}{\text{s}}\right)\left(2.0 \times 10^2\ \frac{\text{kg}}{\text{s}}\right) \\ &= 5 \times 10^4\ \text{N}. \end{aligned}$$

b. Above, we defined m to be the combined mass of the empty rocket plus however much unburned fuel it contained: $m = m_R + m_g$. From Newton's second law,

$$a = \frac{F}{m} = \frac{F}{m_R + m_g}.$$

The force is constant and the empty rocket mass m_R is constant, but the fuel mass m_g is decreasing at a uniform rate; specifically:

$$m_g = m_g(t) = m_{g_0} - \left(\frac{dm_g}{dt}\right)t.$$

This gives us

$$a(t) = \frac{F}{m_{g_\text{i}} - \left(\frac{dm_g}{dt}\right)t} = \frac{F}{M - \left(\frac{dm_g}{dt}\right)t}.$$

Notice that, as expected, the acceleration is a function of time. Substituting the given numbers:

$$a(t) = \frac{5 \times 10^4\ \text{N}}{2.0 \times 10^4\ \text{kg} - \left(2.0 \times 10^2\ \frac{\text{kg}}{\text{s}}\right)t}.$$

c. At $t = 0\,\text{s}$:

$$a(0\ \text{s}) = \frac{5 \times 10^4\ \text{N}}{2.0 \times 10^4\ \text{kg} - \left(2.0 \times 10^2\ \frac{\text{kg}}{\text{s}}\right)(0\ \text{s})} = 2.5\frac{\text{m}}{\text{s}^2}.$$

At $t = 15\ \text{s}$, $a(15\ \text{s}) = 2.9\ \text{m/s}^2$.

At $t = 30\ \text{s}$, $a(30\ \text{s}) = 3.6\ \text{m/s}^2$.

Acceleration is increasing, as we expected.

Significance

Notice that the acceleration is not constant; as a result, any dynamical quantities must be calculated either using integrals, or (more easily) conservation of total energy.

9.14 **Check Your Understanding** What is the physical difference (or relationship) between $\frac{dm}{dt}$ and $\frac{dm_g}{dt}$ in this example?

Rocket in a Gravitational Field

Let's now analyze the velocity change of the rocket during the launch phase, from the surface of Earth. To keep the math manageable, we'll restrict our attention to distances for which the acceleration caused by gravity can be treated as a constant g.

The analysis is similar, except that now there is an external force of $\vec{\mathbf{F}} = -mg\hat{\mathbf{j}}$ acting on our system. This force applies an impulse $d\vec{\mathbf{J}} = \vec{\mathbf{F}}dt = -mgdt\hat{\mathbf{j}}$, which is equal to the change of momentum. This gives us

$$\begin{aligned} d\vec{\mathbf{p}} &= d\vec{\mathbf{J}} \\ \vec{\mathbf{p}}_{\mathrm{f}} - \vec{\mathbf{p}}_{\mathrm{i}} &= -mgdt\hat{\mathbf{j}} \\ \left[(m - dm_g)(v + dv) + dm_g(v - u) - mv\right]\hat{\mathbf{j}} &= -mgdt\hat{\mathbf{j}} \end{aligned}$$

and so

$$mdv - dm_g u = -mgdt$$

where we have again neglected the term $dm_g dv$ and dropped the vector notation. Next we replace dm_g with $-dm$:

$$\begin{aligned} mdv + dmu &= -mgdt \\ mdv &= -dmu - mgdt. \end{aligned}$$

Dividing through by m gives

$$dv = -u\frac{dm}{m} - gdt$$

and integrating, we have

$$\Delta v = u\ln\left(\frac{m_{\mathrm{i}}}{m}\right) - g\Delta t. \tag{9.39}$$

Unsurprisingly, the rocket's velocity is affected by the (constant) acceleration of gravity.

Remember that Δt is the burn time of the fuel. Now, in the absence of gravity, **Equation 9.38** implies that it makes no difference how much time it takes to burn the entire mass of fuel; the change of velocity does not depend on Δt. However, in the presence of gravity, it matters a lot. The $-g\,\Delta t$ term in **Equation 9.39** tells us that the *longer* the burn time is, the *smaller* the rocket's change of velocity will be. This is the reason that the launch of a rocket is so spectacular at the first moment of liftoff: It's essential to burn the fuel as quickly as possible, to get as large a Δv as possible.

CHAPTER 9 REVIEW

KEY TERMS

center of mass weighted average position of the mass

closed system system for which the mass is constant and the net external force on the system is zero

elastic collision that conserves kinetic energy

explosion single object breaks up into multiple objects; kinetic energy is not conserved in explosions

external force force applied to an extended object that changes the momentum of the extended object as a whole

impulse effect of applying a force on a system for a time interval; this time interval is usually small, but does not have to be

impulse-momentum theorem change of momentum of a system is equal to the impulse applied to the system

inelastic collision that does not conserve kinetic energy

internal force force that the simple particles that make up an extended object exert on each other. Internal forces can be attractive or repulsive

Law of Conservation of Momentum total momentum of a closed system cannot change

linear mass density λ, expressed as the number of kilograms of material per meter

momentum measure of the quantity of motion that an object has; it takes into account both how fast the object is moving, and its mass; specifically, it is the product of mass and velocity; it is a vector quantity

perfectly inelastic collision after which all objects are motionless, the final kinetic energy is zero, and the loss of kinetic energy is a maximum

rocket equation derived by the Soviet physicist Konstantin Tsiolkovsky in 1897, it gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass from m_i down to m

system object or collection of objects whose motion is currently under investigation; however, your system is defined at the start of the problem, you must keep that definition for the entire problem

KEY EQUATIONS

Definition of momentum	$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$
Impulse	$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F}(t)dt$ or $\vec{\mathbf{J}} = \vec{\mathbf{F}}_{ave}\Delta t$
Impulse-momentum theorem	$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}$
Average force from momentum	$\vec{\mathbf{F}} = \frac{\Delta\vec{\mathbf{p}}}{\Delta t}$
Instantaneous force from momentum (Newton's second law)	$\vec{\mathbf{F}}(t) = \frac{d\vec{\mathbf{p}}}{dt}$
Conservation of momentum	$\frac{d\vec{\mathbf{p}}_1}{dt} + \frac{d\vec{\mathbf{p}}_2}{dt} = 0$ or $\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 = \text{constant}$
Generalized conservation of momentum	$\sum_{j=1}^{N} \vec{\mathbf{p}}_j = \text{constant}$

Conservation of momentum in two dimensions	$p_{f,\,x} = p_{1,i,\,x} + p_{2,i,\,x}$ $p_{f,\,y} = p_{1,i,\,y} + p_{2,i,\,y}$
External forces	$\vec{\mathbf{F}}_{\text{ext}} = \sum_{j=1}^{N} \frac{d\vec{\mathbf{p}}_j}{dt}$
Newton's second law for an extended object	$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}_{\text{CM}}}{dt}$
Acceleration of the center of mass	$\vec{\mathbf{a}}_{\text{CM}} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{a}}_j$
Position of the center of mass for a system of particles	$\vec{\mathbf{r}}_{\text{CM}} \equiv \frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j$
Velocity of the center of mass	$\vec{\mathbf{v}}_{\text{CM}} = \frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{r}}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{\mathbf{v}}_j$
Position of the center of mass of a continuous object	$\vec{\mathbf{r}}_{\text{CM}} \equiv \frac{1}{M}\int \vec{\mathbf{r}}\, dm$
Rocket equation	$\Delta v = u \ln\left(\frac{m_i}{m}\right)$

SUMMARY

9.1 Linear Momentum

- The motion of an object depends on its mass as well as its velocity. Momentum is a concept that describes this. It is a useful and powerful concept, both computationally and theoretically. The SI unit for momentum is kg · m/s.

9.2 Impulse and Collisions

- When a force is applied on an object for some amount of time, the object experiences an impulse.
- This impulse is equal to the object's change of momentum.
- Newton's second law in terms of momentum states that the net force applied to a system equals the rate of change of the momentum that the force causes.

9.3 Conservation of Linear Momentum

- The law of conservation of momentum says that the momentum of a closed system is constant in time (conserved).
- A closed (or isolated) system is defined to be one for which the mass remains constant, and the net external force is zero.
- The total momentum of a system is conserved *only* when the system is closed.

9.4 Types of Collisions

- An elastic collision is one that conserves kinetic energy.
- An inelastic collision does not conserve kinetic energy.
- Momentum is conserved regardless of whether or not kinetic energy is conserved.
- Analysis of kinetic energy changes and conservation of momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one-dimensional, two-body collisions.

9.5 Collisions in Multiple Dimensions

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes.
- Momentum is conserved in both directions simultaneously and independently.
- The Pythagorean theorem gives the magnitude of the momentum vector using the *x*- and *y*-components, calculated using conservation of momentum in each direction.

9.6 Center of Mass

- An extended object (made up of many objects) has a defined position vector called the center of mass.
- The center of mass can be thought of, loosely, as the average location of the total mass of the object.
- The center of mass of an object traces out the trajectory dictated by Newton's second law, due to the net external force.
- The internal forces within an extended object cannot alter the momentum of the extended object as a whole.

9.7 Rocket Propulsion

- A rocket is an example of conservation of momentum where the mass of the system is not constant, since the rocket ejects fuel to provide thrust.
- The rocket equation gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass.

CONCEPTUAL QUESTIONS

9.1 Linear Momentum

1. An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

2. An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

9.2 Impulse and Collisions

3. Is it possible for a small force to produce a larger impulse on a given object than a large force? Explain.

4. Why is a 10-m fall onto concrete far more dangerous than a 10-m fall onto water?

5. What external force is responsible for changing the momentum of a car moving along a horizontal road?

6. A piece of putty and a tennis ball with the same mass are thrown against a wall with the same velocity. Which object experience a greater impulse from the wall or are the impulses equal? Explain.

9.3 Conservation of Linear Momentum

7. Under what circumstances is momentum conserved?

8. Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

9. Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

10. Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

11. A sprinter accelerates out of the starting blocks. Can you consider him as a closed system? Explain.

12. A rocket in deep space (zero gravity) accelerates by firing hot gas out of its thrusters. Does the rocket constitute a closed system? Explain.

9.4 Types of Collisions

13. Two objects of equal mass are moving with equal and opposite velocities when they collide. Can all the kinetic energy be lost in the collision?

14. Describe a system for which momentum is conserved but mechanical energy is not. Now the reverse: Describe a system for which kinetic energy is conserved but momentum is not.

9.5 Collisions in Multiple Dimensions

15. Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

9.6 Center of Mass

16. Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How does the explosion affect the motion of the center of mass? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

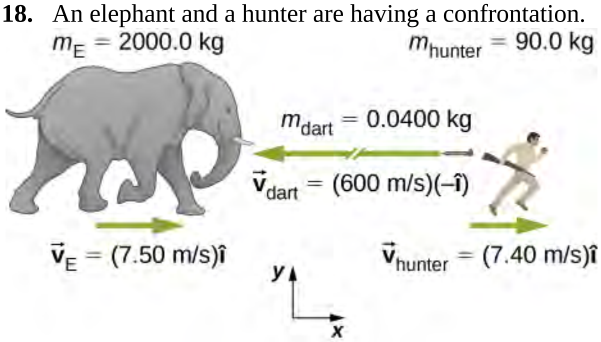
9.7 Rocket Propulsion

17. It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

PROBLEMS

9.1 Linear Momentum

18. An elephant and a hunter are having a confrontation.



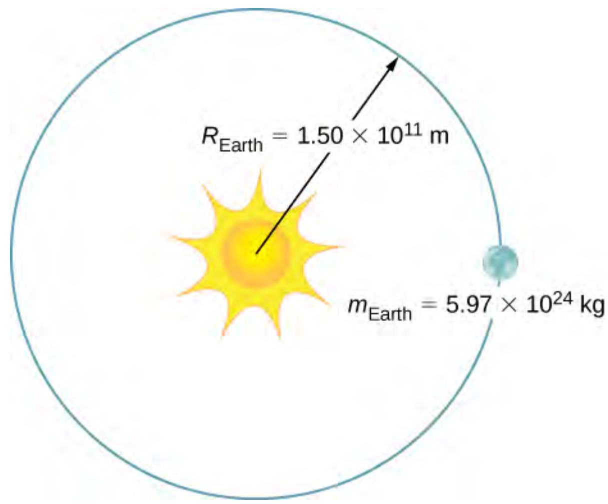
a. Calculate the momentum of the 2000.0-kg elephant charging the hunter at a speed of 7.50 m/s.
b. Calculate the ratio of the elephant's momentum to the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s.
c. What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

19. A skater of mass 40 kg is carrying a box of mass 5 kg. The skater has a speed of 5 m/s with respect to the floor and is gliding without any friction on a smooth surface.

a. Find the momentum of the box with respect to the floor.
b. Find the momentum of the box with respect to the floor after she puts the box down on the frictionless skating surface.

20. A car of mass 2000 kg is moving with a constant velocity of 10 m/s due east. What is the momentum of the car?

21. The mass of Earth is 5.97×10^{24} kg and its orbital radius is an average of 1.50×10^{11} m. Calculate the magnitude of its average linear momentum.



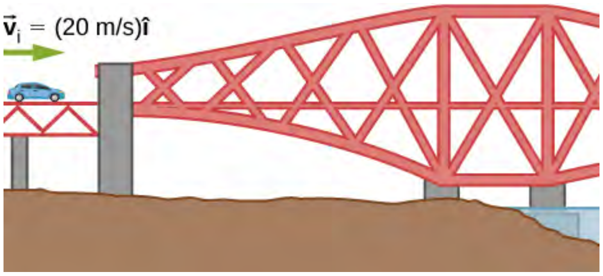
22. If a rainstorm drops 1 cm of rain over an area of 10 km^2 in the period of 1 hour, what is the momentum of the rain that falls in one second? Assume the terminal velocity of a raindrop is 10 m/s.

23. What is the average momentum of an avalanche that moves a 40-cm-thick layer of snow over an area of 100 m by 500 m over a distance of 1 km down a hill in 5.5 s? Assume a density of 350 kg/m^3 for the snow.

24. What is the average momentum of a 70.0-kg sprinter who runs the 100-m dash in 9.65 s?

9.2 Impulse and Collisions

25. A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment (see the following figure).

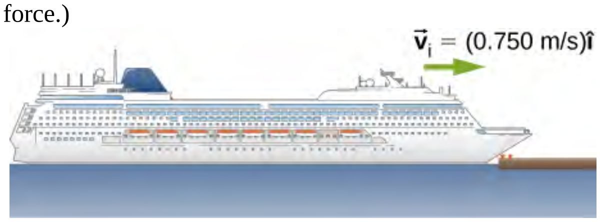


a. Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm.

b. Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

26. One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of 4.00×10^3 m/s, given the collision lasts 6.00×10^{-8} s.

27. A cruise ship with a mass of 1.00×10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest after traveling 6.00 m, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (*Hint*: First calculate the time it took to bring the ship to rest, assuming a constant force.)



28. Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N for 5.50×10^{-2} s.

29. Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

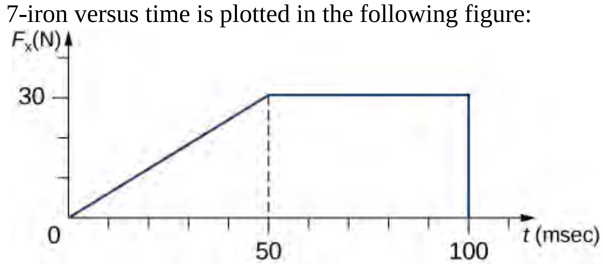
30. A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. Assume constant acceleration of the hammer-nail pair.

a. Calculate the duration of the impact.

b. What was the average force exerted on the nail?

31. What is the momentum (as a function of time) of a 5.0-kg particle moving with a velocity $\vec{v}(t) = \left(2.0\,\hat{i} + 4.0t\,\hat{j}\right)$ m/s? What is the net force acting on this particle?

32. The x-component of a force on a 46-g golf ball by a 7-iron versus time is plotted in the following figure:



a. Find the x-component of the impulse during the intervals

i. [0, 50 ms], and

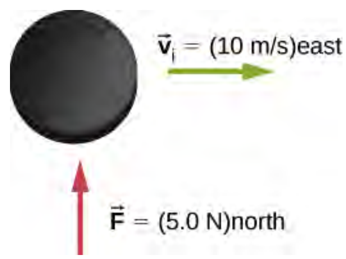
ii. [50 ms, 100 ms]

b. Find the change in the x-component of the momentum during the intervals

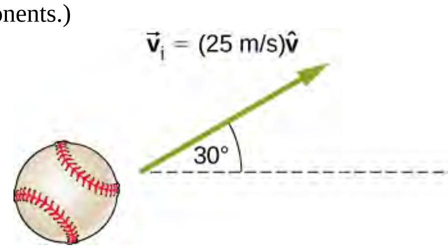
iii. [0, 50 ms], and

iv. [50 ms, 100 ms]

33. A hockey puck of mass 150 g is sliding due east on a frictionless table with a speed of 10 m/s. Suddenly, a constant force of magnitude 5 N and direction due north is applied to the puck for 1.5 s. Find the north and east components of the momentum at the end of the 1.5-s interval.



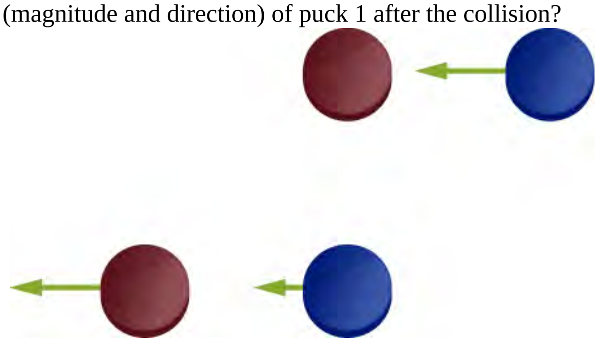
34. A ball of mass 250 g is thrown with an initial velocity of 25 m/s at an angle of $30°$ with the horizontal direction. Ignore air resistance. What is the momentum of the ball after 0.2 s? (Do this problem by finding the components of the momentum first, and then constructing the magnitude and direction of the momentum vector from the components.)



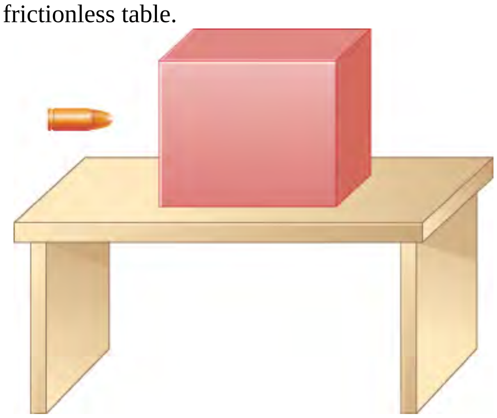
9.3 Conservation of Linear Momentum

35. Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of $1.50 \times 10^5\ \text{kg}$ and a velocity of $(0.30\ \text{m/s})\,\hat{\mathbf{i}}$, and the second having a mass of $1.10 \times 10^5\ \text{kg}$ and a velocity of $-(0.12\ \text{m/s})\,\hat{\mathbf{i}}$. What is their final velocity?

36. Two identical pucks collide elastically on an air hockey table. Puck 1 was originally at rest; puck 2 has an incoming speed of 6.00 m/s and scatters at an angle of $30°$ with respect to its incoming direction. What is the velocity (magnitude and direction) of puck 1 after the collision?



37. The figure below shows a bullet of mass 200 g traveling horizontally towards the east with speed 400 m/s, which strikes a block of mass 1.5 kg that is initially at rest on a frictionless table.



After striking the block, the bullet is embedded in the block and the block and the bullet move together as one unit.

a. What is the magnitude and direction of the velocity of the block/bullet combination immediately after the impact?
b. What is the magnitude and direction of the impulse by the block on the bullet?
c. What is the magnitude and direction of the impulse from the bullet on the block?
d. If it took 3 ms for the bullet to change the speed from 400 m/s to the final speed after impact, what is the average force between the block and the bullet during this time?

38. A 20-kg child is coasting at 3.3 m/s over flat ground in a 4.0-kg wagon. The child drops a 1.0-kg ball out the back of the wagon. What is the final speed of the child and wagon?

39. A 5000-kg paving truck coasts over a road at 2.5 m/s and quickly dumps 1000 kg of gravel on the road. What is the speed of the truck after dumping the gravel?

40. Explain why a cannon recoils when it fires a shell.

41. Two figure skaters are coasting in the same direction, with the leading skater moving at 5.5 m/s and the trailing skating moving at 6.2 m/s. When the trailing skater catches up with the leading skater, he picks her up without applying any horizontal forces on his skates. If the trailing skater is 50% heavier than the 50-kg leading skater, what is their speed after he picks her up?

42. A 2000-kg railway freight car coasts at 4.4 m/s underneath a grain terminal, which dumps grain directly down into the freight car. If the speed of the loaded freight car must not go below 3.0 m/s, what is the maximum mass of grain that it can accept?

9.4 Types of Collisions

43. A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle to the initial direction of the bowling ball and with a speed of 15.0 m/s.

a. Calculate the final velocity (magnitude and direction) of the bowling ball.
b. Is the collision elastic?

44. Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei from gold-197 nuclei. The energy of the incoming helium nucleus was $8.00 \times 10^{-13}\ \text{J}$, and the masses of the helium and gold nuclei were $6.68 \times 10^{-27}\ \text{kg}$ and $3.29 \times 10^{-25}\ \text{kg}$, respectively (note that their mass ratio is 4 to 197).

a. If a helium nucleus scatters to an angle of $120°$ during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus.

b. What is the final kinetic energy of the helium nucleus?

45. A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

46. A 100-g firecracker is launched vertically into the air and explodes into two pieces at the peak of its trajectory. If a 72-g piece is projected horizontally to the left at 20 m/s, what is the speed and direction of the other piece?

47. In an elastic collision, a 400-kg bumper car collides directly from behind with a second, identical bumper car that is traveling in the same direction. The initial speed of the leading bumper car is 5.60 m/s and that of the trailing car is 6.00 m/s. Assuming that the mass of the drivers is much, much less than that of the bumper cars, what are their final speeds?

48. Repeat the preceding problem if the mass of the leading bumper car is 30.0% greater than that of the trailing bumper car.

49. An alpha particle (^{4}He) undergoes an elastic collision with a stationary uranium nucleus (^{235}U). What percent of the kinetic energy of the alpha particle is transferred to the uranium nucleus? Assume the collision is one-dimensional.

50. You are standing on a very slippery icy surface and throw a 1-kg football horizontally at a speed of 6.7 m/s. What is your velocity when you release the football? Assume your mass is 65 kg.

51. A 35-kg child sleds down a hill and then coasts along the flat section at the bottom, where a second 35-kg child jumps on the sled as it passes by her. If the speed of the sled is 3.5 m/s before the second child jumps on, what is its speed after she jumps on?

52. A boy sleds down a hill and onto a frictionless ice-covered lake at 10.0 m/s. In the middle of the lake is a 1000-kg boulder. When the sled crashes into the boulder, he is propelled over the boulder and continues sliding over the ice. If the boy's mass is 40.0 kg and the sled's mass is 2.50 kg, what is the speed of the sled and the boulder after the collision?

9.5 Collisions in Multiple Dimensions

53. A 1.80-kg falcon is diving at 28.0 m/s at a downward angle of $35°$. It catches a 0.650-kg dove from behind in midair. What is their combined velocity after impact if the dove's initial velocity was 7.00 m/s directed horizontally? Note that $\hat{\mathbf{v}}_{1,i}$ is a unit vector pointing in the direction in which the hawk is initially flying.

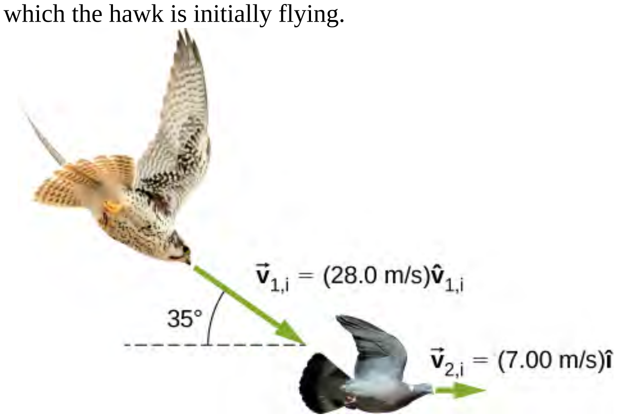
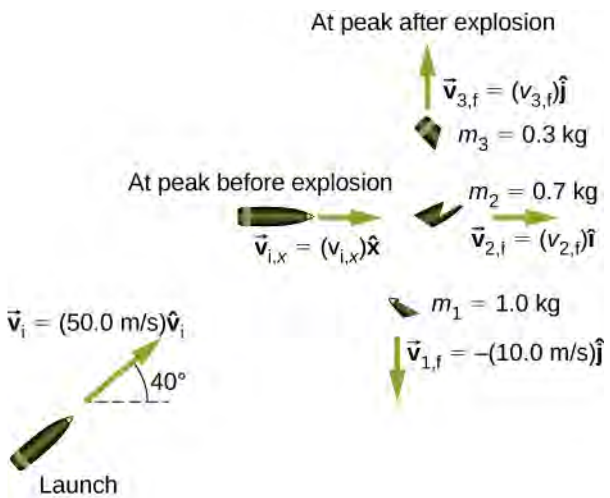


Figure 9.34 (credit "hawk": modification of work by "USFWS Mountain-Prairie"/Flickr; credit "dove": modification of work by Jacob Spinks)

54. A billiard ball, labeled 1, moving horizontally strikes another billiard ball, labeled 2, at rest. Before impact, ball 1 was moving at a speed of 3.00 m/s, and after impact it is moving at 0.50 m/s at 50° from the original direction. If the two balls have equal masses of 300 g, what is the velocity of the ball 2 after the impact?

55. A projectile of mass 2.0 kg is fired in the air at an angle of $40.0°$ to the horizon at a speed of 50.0 m/s. At the highest point in its flight, the projectile breaks into three parts of mass 1.0 kg, 0.7 kg, and 0.3 kg. The 1.0-kg part falls straight down after breakup with an initial speed of 10.0 m/s, the 0.7-kg part moves in the original forward direction, and the 0.3-kg part goes straight up.



a. Find the speeds of the 0.3-kg and 0.7-kg pieces immediately after the break-up.
b. How high from the break-up point does the 0.3-kg piece go before coming to rest?
c. Where does the 0.7-kg piece land relative to where it was fired from?

56. Two asteroids collide and stick together. The first asteroid has mass of 15×10^3 kg and is initially moving at 770 m/s. The second asteroid has mass of 20×10^3 kg and is moving at 1020 m/s. Their initial velocities made an angle of 20° with respect to each other. What is the final speed and direction with respect to the velocity of the first asteroid?

57. A 200-kg rocket in deep space moves with a velocity of $(121\ \text{m/s})\,\hat{\mathbf{i}} + (38.0\ \text{m/s})\,\hat{\mathbf{j}}$. Suddenly, it explodes into three pieces, with the first (78 kg) moving at $-(321\ \text{m/s})\,\hat{\mathbf{i}} + (228\ \text{m/s})\,\hat{\mathbf{j}}$ and the second (56 kg) moving at $(16.0\ \text{m/s})\,\hat{\mathbf{i}} - (88.0\ \text{m/s})\,\hat{\mathbf{j}}$. Find the velocity of the third piece.

58. A proton traveling at 3.0×10^6 m/s scatters elastically from an initially stationary alpha particle and is deflected at an angle of 85° with respect to its initial velocity. Given that the alpha particle has four times the mass of the proton, what percent of its initial kinetic energy does the proton retain after the collision?

59. Three 70-kg deer are standing on a flat 200-kg rock that is on an ice-covered pond. A gunshot goes off and the dear scatter, with deer A running at $(15\ \text{m/s})\,\hat{\mathbf{i}} + (5.0\ \text{m/s})\,\hat{\mathbf{j}}$, deer B running at $(-12\ \text{m/s})\,\hat{\mathbf{i}} + (8.0\ \text{m/s})\,\hat{\mathbf{j}}$, and deer C running at $(1.2\ \text{m/s})\,\hat{\mathbf{i}} - (18.0\ \text{m/s})\,\hat{\mathbf{j}}$. What is the velocity of the rock on which they were standing?

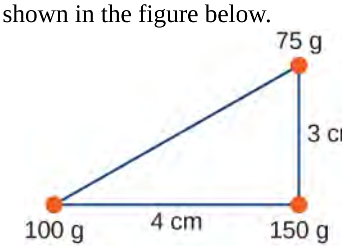
60. A family is skating. The father (75 kg) skates at 8.2 m/s and collides and sticks to the mother (50 kg), who was initially moving at 3.3 m/s and at 45° with respect to the father's velocity. The pair then collides with their daughter (30 kg), who was stationary, and the three slide off together. What is their final velocity?

61. An oxygen atom (mass 16 u) moving at 733 m/s at 15.0° with respect to the $\hat{\mathbf{i}}$ direction collides and sticks to an oxygen molecule (mass 32 u) moving at 528 m/s at 128° with respect to the $\hat{\mathbf{i}}$ direction. The two stick together to form ozone. What is the final velocity of the ozone molecule?

62. Two cars approach an extremely icy four-way perpendicular intersection. Car A travels northward at 30 m/s and car B is travelling eastward. They collide and stick together, traveling at 28° north of east. What was the initial velocity of car B?

9.6 Center of Mass

63. Three point masses are placed at the corners of a triangle as shown in the figure below.



Find the center of mass of the three-mass system.

64. Two particles of masses m_1 and m_2 separated by a horizontal distance D are released from the same height h at the same time. Find the vertical position of the center of mass of these two particles at a time before the two particles strike the ground. Assume no air resistance.

65. Two particles of masses m_1 and m_2 separated by a horizontal distance D are let go from the same height h at different times. Particle 1 starts at $t = 0$, and particle 2 is let go at $t = T$. Find the vertical position of the center of mass at a time before the first particle strikes the ground. Assume no air resistance.

66. Two particles of masses m_1 and m_2 move uniformly in different circles of radii R_1 and R_2 about origin in the *x*, *y*-plane. The *x*- and *y*-coordinates of the center of mass and that of particle 1 are given as follows (where length is in meters and t in seconds):

$x_1(t) = 4\cos(2t),\ y_1(t) = 4\sin(2t)$

and:

$x_{CM}(t) = 3\cos(2t),\ y_{CM}(t) = 3\sin(2t)$.

a. Find the radius of the circle in which particle 1 moves.

b. Find the x- and y-coordinates of particle 2 and the radius of the circle this particle moves.

67. Two particles of masses m_1 and m_2 move uniformly in different circles of radii R_1 and R_2 about the origin in the x, y-plane. The coordinates of the two particles in meters are given as follows ($z = 0$ for both). Here t is in seconds:

$$\begin{aligned} x_1(t) &= 4\cos(2t) \\ y_1(t) &= 4\sin(2t) \\ x_2(t) &= 2\cos\left(3t - \frac{\pi}{2}\right) \\ y_2(t) &= 2\sin\left(3t - \frac{\pi}{2}\right) \end{aligned}$$

a. Find the radii of the circles of motion of both particles.

b. Find the x- and y-coordinates of the center of mass.

c. Decide if the center of mass moves in a circle by plotting its trajectory.

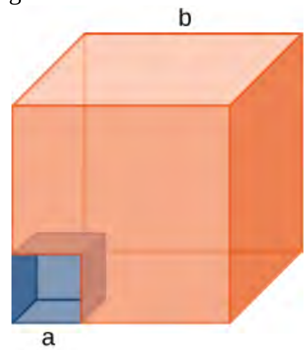
68. Find the center of mass of a one-meter long rod, made of 50 cm of iron (density $8\,\frac{\text{g}}{\text{cm}^3}$) and 50 cm of aluminum (density $2.7\,\frac{\text{g}}{\text{cm}^3}$).

69. Find the center of mass of a rod of length L whose mass density changes from one end to the other quadratically. That is, if the rod is laid out along the x-axis with one end at the origin and the other end at $x = L$, the density is given by $\rho(x) = \rho_0 + (\rho_1 - \rho_0)\left(\frac{x}{L}\right)^2$, where ρ_0 and ρ_1 are constant values.

70. Find the center of mass of a rectangular block of length a and width b that has a nonuniform density such that when the rectangle is placed in the x,y-plane with one corner at the origin and the block placed in the first quadrant with the two edges along the x- and y-axes, the density is given by $\rho(x, y) = \rho_0 x$, where ρ_0 is a constant.

71. Find the center of mass of a rectangular material of length a and width b made up of a material of nonuniform density. The density is such that when the rectangle is placed in the xy-plane, the density is given by $\rho(x, y) = \rho_0 xy$.

72. A cube of side a is cut out of another cube of side b as shown in the figure below.



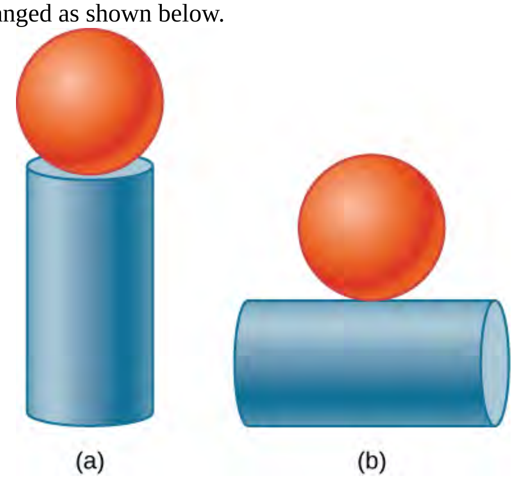
Find the location of the center of mass of the structure. (*Hint:* Think of the missing part as a negative mass overlapping a positive mass.)

73. Find the center of mass of cone of uniform density that has a radius R at the base, height h, and mass M. Let the origin be at the center of the base of the cone and have $+z$ going through the cone vertex.

74. Find the center of mass of a thin wire of mass m and length L bent in a semicircular shape. Let the origin be at the center of the semicircle and have the wire arc from the $+x$ axis, cross the $+y$ axis, and terminate at the $-x$ axis.

75. Find the center of mass of a uniform thin semicircular plate of radius R. Let the origin be at the center of the semicircle, the plate arc from the $+x$ axis to the $-x$ axis, and the z axis be perpendicular to the plate.

76. Find the center of mass of a sphere of mass M and radius R and a cylinder of mass m, radius r, and height h arranged as shown below.



Express your answers in a coordinate system that has the origin at the center of the cylinder.

9.7 Rocket Propulsion

77. (a) A 5.00-kg squid initially at rest ejects 0.250 kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-*N* frictional force opposing the squid's movement?

(b) How much energy is lost to work done against friction?

78. A rocket takes off from Earth and reaches a speed of 100 m/s in 10.0 s. If the exhaust speed is 1500 m/s and the mass of fuel burned is 100 kg, what was the initial mass of the rocket?

79. Repeat the preceding problem but for a rocket that takes off from a space station, where there is no gravity other than the negligible gravity due to the space station.

80. How much fuel would be needed for a 1000-kg rocket (this is its mass with no fuel) to take off from Earth and reach 1000 m/s in 30 s? The exhaust speed is 1000 m/s.

81. What exhaust speed is required to accelerate a rocket in deep space from 800 m/s to 1000 m/s in 5.0 s if the total rocket mass is 1200 kg and the rocket only has 50 kg of fuel left?

82. Unreasonable Results Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water.

(a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0°, assuming negligible lift from the air and negligible air resistance.

(b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected.

(c) What is unreasonable about the results?

(d) Which premise is unreasonable, or which premises are inconsistent?

ADDITIONAL PROBLEMS

83. Two 70-kg canoers paddle in a single, 50-kg canoe. Their paddling moves the canoe at 1.2 m/s with respect to the water, and the river they're in flows at 4 m/s with respect to the land. What is their momentum with respect to the land?

84. Which has a larger magnitude of momentum: a 3000-kg elephant moving at 40 km/h or a 60-kg cheetah moving at 112 km/h?

85. A driver applies the brakes and reduces the speed of her car by 20%, without changing the direction in which the car is moving. By how much does the car's momentum change?

86. You friend claims that momentum is mass multiplied by velocity, so things with more mass have more momentum. Do you agree? Explain.

87. Dropping a glass on a cement floor is more likely to break the glass than if it is dropped from the same height on a grass lawn. Explain in terms of the impulse.

88. Your 1500-kg sports car accelerates from 0 to 30 m/s in 10 s. What average force is exerted on it during this acceleration?

89. A ball of mass m is dropped. What is the formula for the impulse exerted on the ball from the instant it is dropped to an arbitrary time τ later? Ignore air resistance.

90. Repeat the preceding problem, but including a drag force due to air of $f_{\text{drag}} = -b\,\vec{v}$.

91. A 5.0-g egg falls from a 90-cm-high counter onto the floor and breaks. What impulse is exerted by the floor on the egg?

92. A car crashes into a large tree that does not move. The car goes from 30 m/s to 0 in 1.3 m. (a) What impulse is applied to the driver by the seatbelt, assuming he follows the same motion as the car? (b) What is the average force applied to the driver by the seatbelt?

93. Two hockey players approach each other head on, each traveling at the same speed v_{i} . They collide and get tangled together, falling down and moving off at a speed $v_{\text{i}}/5$. What is the ratio of their masses?

94. You are coasting on your 10-kg bicycle at 15 m/s and a 5.0-g bug splatters on your helmet. The bug was initially moving at 2.0 m/s in the same direction as you. If your mass is 60 kg, (a) what is the initial momentum of you plus your bicycle? (b) What is the initial momentum of the bug? (c) What is your change in velocity due to the collision with the bug? (d) What would the change in velocity have been if the bug were traveling in the opposite direction?

95. A load of gravel is dumped straight down into a 30 000-kg freight car coasting at 2.2 m/s on a straight section of a railroad. If the freight car's speed after receiving the gravel is 1.5 m/s, what mass of gravel did it receive?

96. Two carts on a straight track collide head on. The first cart was moving at 3.6 m/s in the positive *x* direction and the second was moving at 2.4 m/s in the opposite direction. After the collision, the second car continues moving in its initial direction of motion at 0.24 m/s. If the mass of the second car is 5.0 times that of the first, what is the mass and final velocity of the first car?

97. A 100-kg astronaut finds himself separated from his spaceship by 10 m and moving away from the spaceship at 0.1 m/s. To get back to the spaceship, he throws a 10-kg tool bag away from the spaceship at 5.0 m/s. How long will he take to return to the spaceship?

98. Derive the equations giving the final speeds for two objects that collide elastically, with the mass of the objects being m_1 and m_2 and the initial speeds being $v_{1,\text{i}}$ and $v_{2,\text{i}} = 0$ (i.e., second object is initially stationary).

99. Repeat the preceding problem for the case when the initial speed of the second object is nonzero.

100. A child sleds down a hill and collides at 5.6 m/s into a stationary sled that is identical to his. The child is launched forward at the same speed, leaving behind the two sleds that lock together and slide forward more slowly. What is the speed of the two sleds after this collision?

101. For the preceding problem, find the final speed of each sled for the case of an elastic collision.

102. A 90-kg football player jumps vertically into the air to catch a 0.50-kg football that is thrown essentially horizontally at him at 17 m/s. What is his horizontal speed after catching the ball?

103. Three skydivers are plummeting earthward. They are initially holding onto each other, but then push apart. Two skydivers of mass 70 and 80 kg gain horizontal velocities of 1.2 m/s north and 1.4 m/s southeast, respectively. What is the horizontal velocity of the third skydiver, whose mass is 55 kg?

104. Two billiard balls are at rest and touching each other on a pool table. The cue ball travels at 3.8 m/s along the line of symmetry between these balls and strikes them simultaneously. If the collision is elastic, what is the velocity of the three balls after the collision?

105. A billiard ball traveling at $(2.2\text{ m/s})\hat{\mathbf{i}} - (0.4\text{ m/s})\hat{\mathbf{j}}$ collides with a wall that is aligned in the $\hat{\mathbf{j}}$ direction. Assuming the collision is elastic, what is the final velocity of the ball?

106. Two identical billiard balls collide. The first one is initially traveling at $(2.2\text{ m/s})\hat{\mathbf{i}} - (0.4\text{ m/s})\hat{\mathbf{j}}$ and the second one at $-(1.4\text{ m/s})\hat{\mathbf{i}} + (2.4\text{ m/s})\hat{\mathbf{j}}$. Suppose they collide when the center of ball 1 is at the origin and the center of ball 2 is at the point $(2R,\ 0)$ where *R* is the radius of the balls. What is the final velocity of each ball?

107. Repeat the preceding problem if the balls collide when the center of ball 1 is at the origin and the center of ball 2 is at the point $(0,\ 2R)$.

108. Repeat the preceding problem if the balls collide when the center of ball 1 is at the origin and the center of ball 2 is at the point $\left(\sqrt{3}R/2,\ R/2\right)$

109. Where is the center of mass of a semicircular wire of radius *R* that is centered on the origin, begins and ends on the *x* axis, and lies in the *x*,*y* plane?

110. Where is the center of mass of a slice of pizza that was cut into eight equal slices? Assume the origin is at the apex of the slice and measure angles with respect to an edge of the slice. The radius of the pizza is *R*.

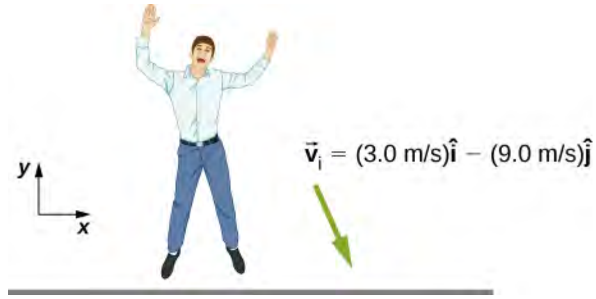
111. If the entire population of Earth were transferred to the Moon, how far would the center of mass of the Earth-Moon-population system move? Assume the population is 7 billion, the average human has a mass of 65 kg, and that the population is evenly distributed over both the Earth and the Moon. The mass of the Earth is $5.97 \times 10^{24}\,\text{kg}$ and that of the Moon is $7.34 \times 10^{22}\,\text{kg}$. The radius of the Moon's orbit is about $3.84 \times 10^{5}\,\text{m}$.

112. You friend wonders how a rocket continues to climb into the sky once it is sufficiently high above the surface of Earth so that its expelled gasses no longer push on the surface. How do you respond?

113. To increase the acceleration of a rocket, should you throw rocks out of the front window of the rocket or out of the back window?

CHALLENGE PROBLEMS

114. A 65-kg person jumps from the first floor window of a burning building and lands almost vertically on the ground with a horizontal velocity of 3 m/s and vertical velocity of -9 m/s. Upon impact with the ground he is brought to rest in a short time. The force experienced by his feet depends on whether he keeps his knees stiff or bends them. Find the force on his feet in each case.



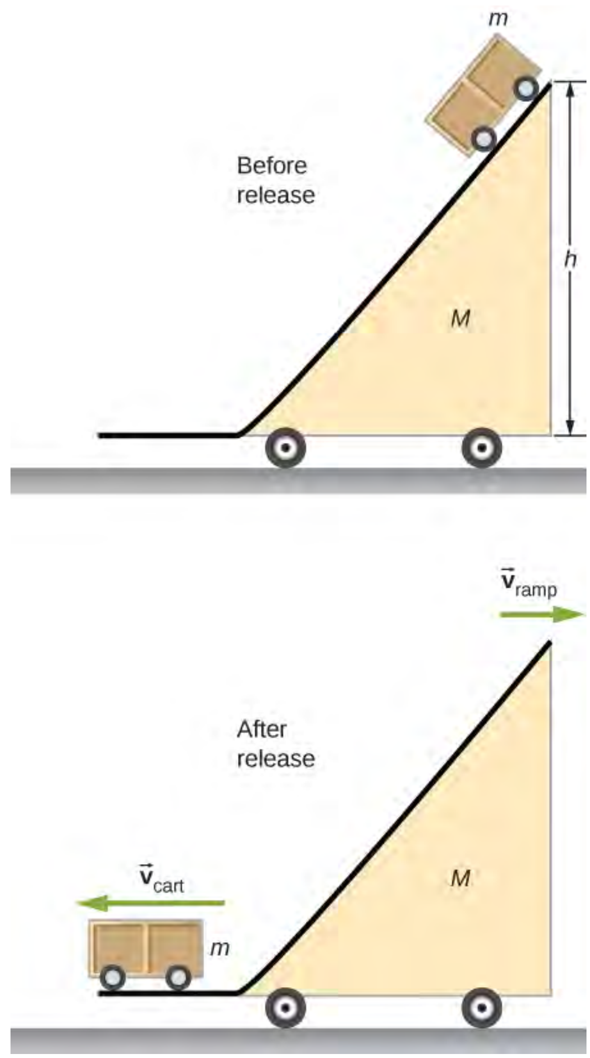
a. First find the impulse on the person from the impact on the ground. Calculate both its magnitude and direction.
b. Find the average force on the feet if the person keeps his leg stiff and straight and his center of mass drops by only 1 cm vertically and 1 cm horizontally during the impact.
c. Find the average force on the feet if the person bends his legs throughout the impact so that his center of mass drops by 50 cm vertically and 5 cm horizontally during the impact.
d. Compare the results of part (b) and (c), and draw conclusions about which way is better.

You will need to find the time the impact lasts by making reasonable assumptions about the deceleration. Although the force is not constant during the impact, working with constant average force for this problem is acceptable.

115. Two projectiles of mass m_1 and m_2 are fired at the same speed but in opposite directions from two launch sites separated by a distance *D*. They both reach the same spot in their highest point and strike there. As a result of the impact they stick together and move as a single body afterwards. Find the place they will land.

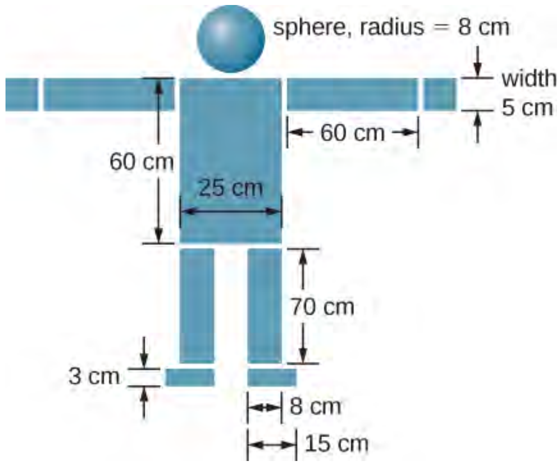
116. Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

117. A ramp of mass *M* is at rest on a horizontal surface. A small cart of mass *m* is placed at the top of the ramp and released.



What are the velocities of the ramp and the cart relative to the ground at the instant the cart leaves the ramp?

118. Find the center of mass of the structure given in the figure below. Assume a uniform thickness of 20 cm, and a uniform density of 1 g/cm^3.


sphere, radius = 8 cm
width
5 cm
60 cm
60 cm
25 cm
70 cm
3 cm
8 cm
15 cm

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7 WORK, ENERGY, AND ENERGY RESOURCES

Figure 7.1 How many forms of energy can you identify in this photograph of a wind farm in Iowa? (credit: Jürgen from Sandesneben, Germany, Wikimedia Commons)

Chapter Outline

Connection for AP® Courses

Energy plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods to the energy we use to run our cars and the sunlight that warms us on the beach. You can also cite examples of what people call “energy” that may not be scientific, such as someone having an energetic personality. Not only does energy have many interesting forms, it is involved in almost all phenomena, and is one of the most important concepts of physics.

There is no simple and accurate scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define energy as the ability to do **work**, admitting that in some circumstances not all energy is available to do work. Because of the association of energy with work, we begin the chapter with a discussion of work. Work is intimately related to energy and how energy moves from one system to another or changes form. The work-energy theorem supports Big Idea 3, that interactions between objects are described by forces. In particular, exerting a force on an object may do work on it, changing it's energy (Enduring Understanding 3.E). The work-energy theorem, introduced in this chapter, establishes the relationship between work done on an object by an external force and changes in the object’s kinetic energy (Essential Knowledge 3.E.1).

Similarly, systems can do work on each other, supporting Big Idea 4, that interactions between systems can result in changes in those systems—in this case, changes in the total energy of the system (Enduring Understanding 4.C). The total energy of the system is the sum of its kinetic energy, potential energy, and microscopic internal energy (Essential Knowledge 4.C.1). In this chapter students learn how to calculate kinetic, gravitational, and elastic potential energy in order to determine the total mechanical energy of a system. The transfer of mechanical energy into or out of a system is equal to the work done on the system by an external force with a nonzero component parallel to the displacement (Essential Knowledge 4.C.2).

An important aspect of energy is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is "conserved." **Conservation of energy** (as physicists call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he suggested that mass is equivalent to energy (his famous equation $E = mc^2$). This is one of the most important applications of Big Idea 5, that changes that occur as a result of interactions are constrained by conservation laws. Specifically, there are many situations where conservation of energy (Enduring Understanding 5.B) is both a useful concept and starting point for calculations related to the system. Note, however, that conservation doesn't necessarily mean that energy in a system doesn't change. Energy may be transferred into or out of the system, and the change must be equal to the amount transferred (Enduring Understanding 5.A). This may occur if there is an external force or a transfer between external objects and the system (Essential Knowledge 5.A.3). Energy is one of the fundamental quantities that are conserved for all systems (Essential Knowledge 5.A.2). The chapter introduces concepts of kinetic energy and potential energy. Kinetic energy is introduced as an energy of motion that can be changed by the amount of work done by an external force. Potential energy can only exist when objects interact with each other via conservative forces according to classical physics (Essential Knowledge 5.B.3). Because of this, a single object can only have kinetic energy and no potential energy (Essential Knowledge 5.B.1). The chapter also introduces the idea that the energy transfer is equal to the work done on the system by external forces and the rate of energy transfer is defined as power (Essential Knowledge 5.B.5).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally. We will briefly examine the world's energy use patterns at the end of this chapter.

The concepts in this chapter support:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.E A force exerted on an object can change the kinetic energy of the object.

Essential Knowledge 3.E.1 The change in the kinetic energy of an object depends on the force exerted on the object and on the displacement of the object during the interval that the force is exerted.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.C Interactions with other objects or systems can change the total energy of a system.

Essential Knowledge 4.C.1 The energy of a system includes its kinetic energy, potential energy, and microscopic internal energy. Examples should include gravitational potential energy, elastic potential energy, and kinetic energy.

Essential Knowledge 4.C.2 Mechanical energy (the sum of kinetic and potential energy) is transferred into or out of a system when an external force is exerted on a system such that a component of the force is parallel to its displacement. The process through which the energy is transferred is called work.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Essential Knowledge 5.A.2 For all systems under all circumstances, energy, charge, linear momentum, and angular momentum are conserved.

Essential Knowledge 5.A.3 An interaction can be either a force exerted by objects outside the system or the transfer of some quantity with objects outside the system.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.1 Classically, an object can only have kinetic energy since potential energy requires an interaction between two or more objects.

Essential Knowledge 5.B.3 A system with internal structure can have potential energy. Potential energy exists within a system if the objects within that system interact with conservative forces.

Essential Knowledge 5.B.5 Energy can be transferred by an external force exerted on an object or system that moves the object or system through a distance; this energy transfer is called work. Energy transfer in mechanical or electrical systems may occur at different rates. Power is defined as the rate of energy transfer into, out of, or within a system.

7.1 Work: The Scientific Definition

Learning Objectives

By the end of this section, you will be able to:

- Explain how an object must be displaced for a force on it to do work.
- Explain how relative directions of force and displacement of an object determine whether the work done on the object is positive, negative, or zero.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.B.5.1** The student is able to design an experiment and analyze data to examine how a force exerted on an object or system does work on the object or system as it moves through a distance. **(S.P. 4.2, 5.1)**

- **5.B.5.2** The student is able to design an experiment and analyze graphical data in which interpretations of the area under a force-distance curve are needed to determine the work done on or by the object or system. **(S.P. 4.5, 5.1)**
- **5.B.5.3** The student is able to predict and calculate from graphical data the energy transfer to or work done on an object or system from information about a force exerted on the object or system through a distance. **(S.P. 1.5, 2.2, 6.4)**

What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done on an object, a force must be exerted on that object and there must be displacement of that object in the direction of the force.

Formally, the **work** done on a system by a constant force is defined to be *the product of the component of the force in the direction of motion and the distance through which the force acts*. For a constant force, this is expressed in equation form as

$$W = \mid \mathbf{F} \mid (\cos\theta) \mid \mathbf{d} \mid , \tag{7.1}$$

where W is work, $\mathbf{d}$ is the displacement of the system, and θ is the angle between the force vector $\mathbf{F}$ and the displacement vector $\mathbf{d}$, as in **Figure 7.2**. We can also write this as

$$W = Fd\cos\theta. \tag{7.2}$$

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.

What is Work?

The work done on a system by a constant force is *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

$$W = Fd\cos\theta, \tag{7.3}$$

where W is work, F is the magnitude of the force on the system, d is the magnitude of the displacement of the system, and θ is the angle between the force vector $\mathbf{F}$ and the displacement vector $\mathbf{d}$.

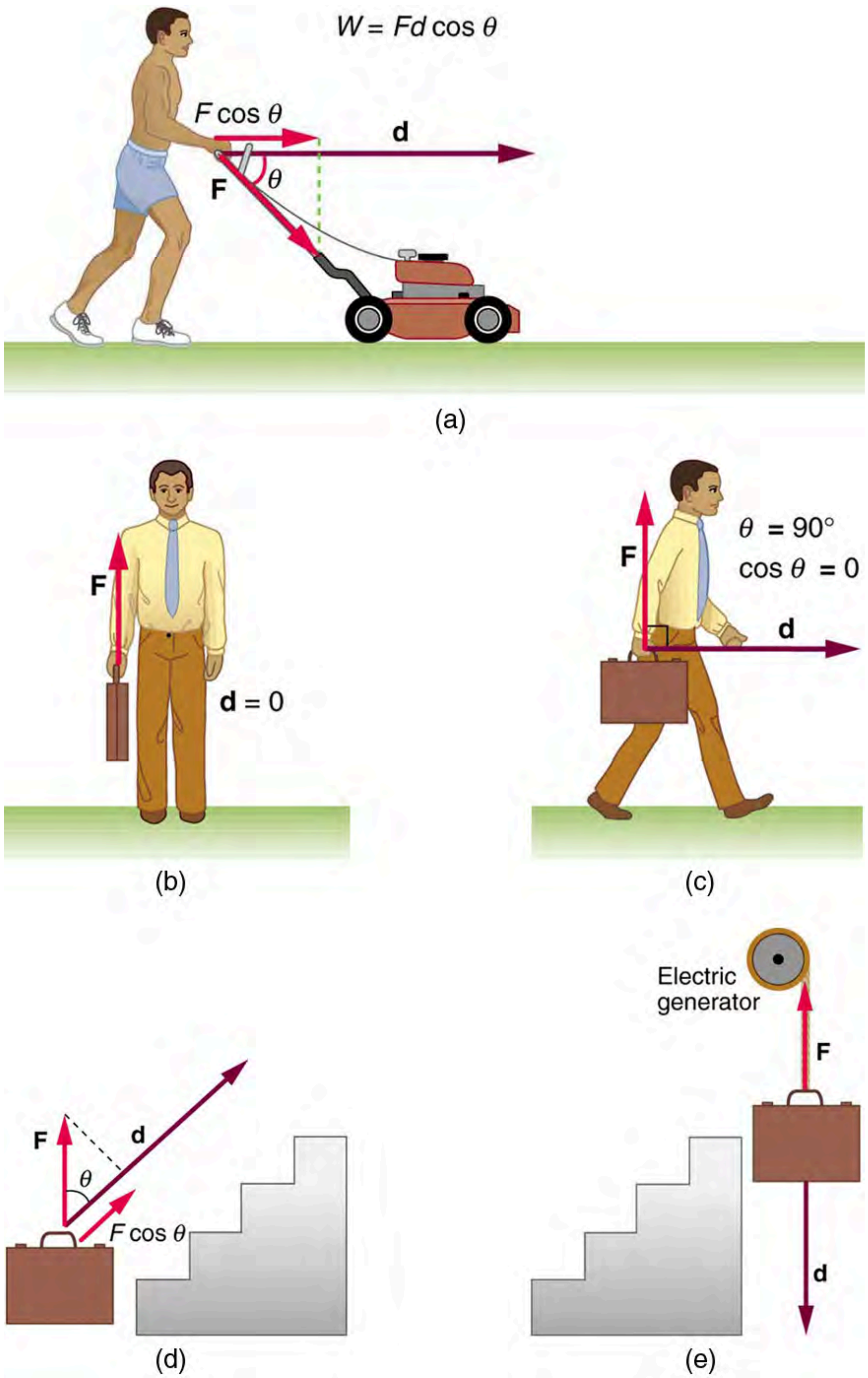

Figure 7.2 Examples of work. (a) The work done by the force $\mathbf{F}$ on this lawn mower is $Fd \cos \theta$. Note that $F \cos \theta$ is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work *is* done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force $\mathbf{F}$ in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because $\mathbf{F}$ and $\mathbf{d}$ are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in **Figure 7.2**. The person holding the briefcase in **Figure 7.2**(b) does no work, for example. Here $d = 0$, so $W = 0$. Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, *but they are doing no work on the system of interest* (the "briefcase-Earth system"—see **Gravitational Potential Energy** for more details). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase on level ground in **Figure 7.2**(c) does no work on it, because the force is perpendicular to the motion. That is, $\cos 90^\circ = 0$, and so $W = 0$.

In contrast, when a force exerted on the system has a component in the direction of motion, such as in **Figure 7.2**(d), work *is* done—energy is transferred to the briefcase. Finally, in **Figure 7.2**(e), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase's weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward on the briefcase, and the displacement downward. This makes $\theta = 180^\circ$, and $\cos 180^\circ = -1$; therefore, W is negative.

Real World Connections: When Work Happens

Note that work as we define it is not the same as effort. You can push against a concrete wall all you want, but you won't move it. While the pushing represents effort on your part, the fact that you have not changed the wall's state in any way indicates that you haven't done work. If you did somehow push the wall over, this would indicate a change in the wall's state, and therefore you would have done work.

This can also be shown with **Figure 7.2**(a): as you push a lawnmower against friction, both you and friction are changing the lawnmower's state. However, only the component of the force parallel to the movement is changing the lawnmower's state. The component perpendicular to the motion is trying to push the lawnmower straight into Earth; the lawnmower does not move into Earth, and therefore the lawnmower's state is not changing in the direction of Earth.

Similarly, in **Figure 7.2**(c), both your hand and gravity are exerting force on the briefcase. However, they are both acting perpendicular to the direction of motion, hence they are not changing the condition of the briefcase and do no work. However, if the briefcase were dropped, then its displacement would be parallel to the force of gravity, which would do work on it, changing its state (it would fall to the ground).

Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in **newton-meters**. A newton-meter is given the special name **joule** (J), and $1\ \text{J} = 1\ \text{N} \cdot \text{m} = 1\ \text{kg} \cdot \text{m}^2/\text{s}^2$. One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

Example 7.1 Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in **Figure 7.2**(a) if he exerts a constant force of $75.0\ \text{N}$ at an angle 35° below the horizontal and pushes the mower $25.0\ \text{m}$ on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of $10{,}000\ \text{kJ}$ (about $2400\ \text{kcal}$) of food energy. One *calorie* (1 cal) of heat is the amount required to warm 1 g of water by 1°C, and is equivalent to $4.184\ \text{J}$, while one *food calorie* (1 kcal) is equivalent to $4184\ \text{J}$.

Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation $W = Fd \cos \theta$. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$$W = Fd \cos \theta. \tag{7.4}$$

Substituting the known values gives

$$\begin{aligned} W &= (75.0\ \text{N})(25.0\ \text{m}) \cos (35.0^\circ) \\ &= 1536\ \text{J} = 1.54 \times 10^3\ \text{J}. \end{aligned} \tag{7.5}$$

Converting the work in joules to kilocalories yields $W = (1536\ \text{J})(1\ \text{kcal}/4184\ \text{J}) = 0.367\ \text{kcal}$. The ratio of the work done to the daily consumption is

$$\frac{W}{2400\ \text{kcal}} = 1.53 \times 10^{-4}. \tag{7.6}$$

Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we "work" all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

Applying the Science Practices: Boxes on Floors

Plan and design an experiment to determine how much work you do on a box when you are pushing it over different floor surfaces. Make sure your experiment can help you answer the following questions: What happens on different surfaces? What happens if you take different routes across the same surface? Do you get different results with two people pushing on perpendicular surfaces of the box? What if you vary the mass in the box? Remember to think about both your effort in any given instant (a proxy for force exerted) and the total work you do. Also, when planning your experiments, remember that in any given set of trials you should only change one variable.

You should find that you have to exert more effort on surfaces that will create more friction with the box, though you might be surprised by which surfaces the box slides across easily. Longer routes result in your doing more work, even though the box ends up in the same place. Two people pushing on perpendicular sides do less work for their total effort, due to the forces and displacement not being parallel. A more massive box will take more effort to move.

Applying the Science Practices: Force-Displacement Diagrams

Suppose you are given two carts and a track to run them on, a motion detector, a force sensor, and a computer that can record the data from the two sensors. Plan and design an experiment to measure the work done on one of the carts, and compare your results to the work-energy theorem. Note that the motion detector can measure both displacement and velocity versus time, while the force sensor measures force over time, and the carts have known masses. Recall that the work-energy theorem states that the work done on a system (force over displacement) should equal the change in kinetic energy. In your experimental design, describe and compare two possible ways to calculate the work done.

Sample Response: One possible technique is to set up the motion detector at one end of the track, and have the computer record both displacement and velocity over time. Then attach the force sensor to one of the carts, and use this cart, through the force sensor, to push the second cart toward the motion detector. Calculate the difference between the final and initial kinetic energies (the kinetic energies after and before the push), and compare this to the area of a graph of force versus displacement for the duration of the push. They should be the same.

7.2 Kinetic Energy and the Work-Energy Theorem

Learning Objectives

By the end of this section, you will be able to:

- Explain work as a transfer of energy and net work as the work done by the net force.
- Explain and apply the work-energy theorem.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.E.1.1** The student is able to make predictions about the changes in kinetic energy of an object based on considerations of the direction of the net force on the object as the object moves. **(S.P. 6.4, 7.2)**
- **3.E.1.2** The student is able to use net force and velocity vectors to determine qualitatively whether kinetic energy of an object would increase, decrease, or remain unchanged. **(S.P. 1.4)**
- **3.E.1.3** The student is able to use force and velocity vectors to determine qualitatively or quantitatively the net force exerted on an object and qualitatively whether kinetic energy of that object would increase, decrease, or remain unchanged. **(S.P. 1.4, 2.2)**
- **3.E.1.4** The student is able to apply mathematical routines to determine the change in kinetic energy of an object given the forces on the object and the displacement of the object. **(S.P. 2.2)**
- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the calculation of component types of energy within the system whose sum is the total energy. **(S.P. 1.4, 2.1, 2.2)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**
- **4.C.2.2** The student is able to apply the concepts of conservation of energy and the work-energy theorem to determine qualitatively and/or quantitatively that work done on a two-object system in linear motion will change the kinetic energy of the center of mass of the system, the potential energy of the systems, and/or the internal energy of the system. **(S.P. 1.4, 2.2, 7.2)**
- **5.B.5.3** The student is able to predict and calculate from graphical data the energy transfer to or work done on an object or system from information about a force exerted on the object or system through a distance. **(S.P. 1.5, 2.2, 6.4)**

Work Transfers Energy

What happens to the work done on a system? Energy is transferred into the system, but in what form? Does it remain in the system or move on? The answers depend on the situation. For example, if the lawn mower in **Figure 7.2**(a) is pushed just hard enough to keep it going at a constant speed, then energy put into the mower by the person is removed continuously by friction, and eventually leaves the system in the form of heat transfer. In contrast, work done on the briefcase by the person carrying it up stairs in **Figure 7.2**(d) is stored in the briefcase-Earth system and can be recovered at any time, as shown in **Figure 7.2**(e). In fact, the building of the pyramids in ancient Egypt is an example of storing energy in a system by doing work on the system.

Some of the energy imparted to the stone blocks in lifting them during construction of the pyramids remains in the stone-Earth system and has the potential to do work.

In this section we begin the study of various types of work and forms of energy. We will find that some types of work leave the energy of a system constant, for example, whereas others change the system in some way, such as making it move. We will also develop definitions of important forms of energy, such as the energy of motion.

Net Work and the Work-Energy Theorem

We know from the study of Newton's laws in **Dynamics: Force and Newton's Laws of Motion** that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work done by all external forces—that is, **net work** is the work done by the net external force $\mathbf{F}_{\text{net}}$. In equation form, this is

$W_{\text{net}} = F_{\text{net}} d \cos \theta$ where θ is the angle between the force vector and the displacement vector.

Figure 7.3(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an $F \cos \theta$ vs. d graph. In this case, $F \cos \theta$ is constant. You can see that the area under the graph is $Fd \cos \theta$, or the work done. **Figure 7.3**(b) shows a more general process where the force varies. The area under the curve is divided into strips, each having an average force $(F \cos \theta)_{i(\text{ave})}$. The work done is $(F \cos \theta)_{i(\text{ave})} d_i$ for each strip, and the total work done is the sum of the W_i. Thus the total work done is the total area under the curve, a useful property to which we shall refer later.

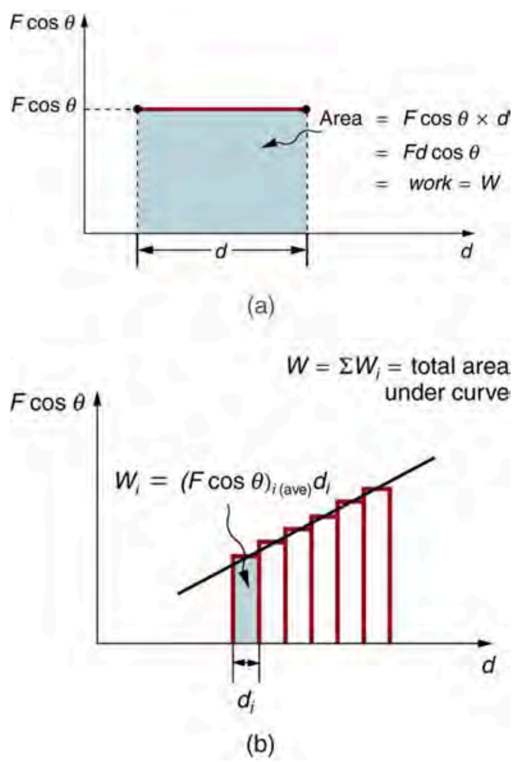

Figure 7.3 (a) A graph of $F \cos \theta$ vs. d, when $F \cos \theta$ is constant. The area under the curve represents the work done by the force. (b) A graph of $F \cos \theta$ vs. d in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.

Real World Connections: Work and Direction

Consider driving in a car. While moving, you have forward velocity and therefore kinetic energy. When you hit the brakes, they exert a force opposite to your direction of motion (acting through the wheels). The brakes do work on your car and reduce the kinetic energy. Similarly, when you accelerate, the engine (acting through the wheels) exerts a force in the direction of motion. The engine does work on your car, and increases the kinetic energy. Finally, if you go around a corner at a constant speed, you have the same kinetic energy both before and after the corner. The force exerted by the engine was perpendicular to the direction of motion, and therefore did no work and did not change the kinetic energy.

Net work will be simpler to examine if we consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in **Figure**

7.4.

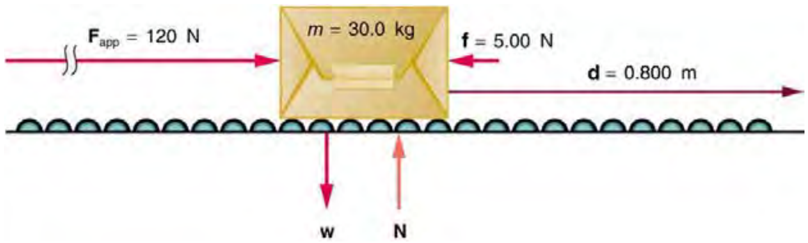

Figure 7.4 A package on a roller belt is pushed horizontally through a distance $\mathbf{d}$.

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force arises solely from the horizontal applied force $\mathbf{F}_{\text{app}}$ and the horizontal friction force $\mathbf{f}$. Thus, as expected, the net force is parallel to the displacement, so that $\theta = 0^{\mathrm{o}}$ and $\cos\theta = 1$, and the net work is given by

$$W_{\text{net}} = F_{\text{net}}d. \tag{7.7}$$

The effect of the net force $\mathbf{F}_{\text{net}}$ is to accelerate the package from v_0 to v . The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See **Example 7.2**.) By using Newton's second law, and doing some algebra, we can reach an interesting conclusion. Substituting $F_{\text{net}} = ma$ from Newton's second law gives

$$W_{\text{net}} = mad. \tag{7.8}$$

To get a relationship between net work and the speed given to a system by the net force acting on it, we take $d = x - x_0$ and use the equation studied in **Motion Equations for Constant Acceleration in One Dimension** for the change in speed over a distance d if the acceleration has the constant value a ; namely, $v^2 = v_0{}^2 + 2ad$ (note that a appears in the expression for the net work). Solving for acceleration gives $a = \dfrac{v^2 - v_0{}^2}{2d}$. When a is substituted into the preceding expression for W_{net} , we obtain

$$W_{\text{net}} = m\left(\frac{v^2 - v_0{}^2}{2d}\right)d. \tag{7.9}$$

The d cancels, and we rearrange this to obtain

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2. \tag{7.10}$$

This expression is called the **work-energy theorem**, and it actually applies *in general* (even for forces that vary in direction and magnitude), although we have derived it for the special case of a constant force parallel to the displacement. The theorem implies that the net work on a system equals the change in the quantity $\frac{1}{2}mv^2$. This quantity is our first example of a form of energy.

The Work-Energy Theorem

The net work on a system equals the change in the quantity $\frac{1}{2}mv^2$.

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 \tag{7.11}$$

The quantity $\frac{1}{2}mv^2$ in the work-energy theorem is defined to be the translational **kinetic energy** (KE) of a mass m moving at a speed v . (*Translational* kinetic energy is distinct from *rotational* kinetic energy, which is considered later.) In equation form, the translational kinetic energy,

$$\text{KE} = \frac{1}{2}mv^2, \tag{7.12}$$

is the energy associated with translational motion. Kinetic energy is a form of energy associated with the motion of a particle, single body, or system of objects moving together.

We are aware that it takes energy to get an object, like a car or the package in **Figure 7.4**, up to speed, but it may be a bit surprising that kinetic energy is proportional to speed squared. This proportionality means, for example, that a car traveling at 100 km/h has four times the kinetic energy it has at 50 km/h, helping to explain why high-speed collisions are so devastating. We will now consider a series of examples to illustrate various aspects of work and energy.

Applying the Science Practices: Cars on a Hill

Assemble a ramp suitable for rolling some toy cars up or down. Then plan a series of experiments to determine how the direction of a force relative to the velocity of an object alters the kinetic energy of the object. Note that gravity will be pointing down in all cases. What happens if you start the car at the top? How about at the bottom, with an initial velocity that is increasing? If your ramp is wide enough, what happens if you send the toy car straight across? Does varying the surface of the ramp change your results?

Sample Response: When the toy car is going down the ramp, with a component of gravity in the same direction, the kinetic energy increases. Sending the car up the ramp decreases the kinetic energy, as gravity is opposing the motion. Sending the car sideways should result in little to no change. If you have a surface that generates more friction than a smooth surface (carpet), note that the friction always opposed the motion, and hence decreases the kinetic energy.

Example 7.2 Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in **Figure 7.4** is moving at 0.500 m/s. What is its kinetic energy?

Strategy

Because the mass m and speed v are given, the kinetic energy can be calculated from its definition as given in the equation $\text{KE} = \frac{1}{2}mv^2$.

Solution

The kinetic energy is given by

$$\text{KE} = \frac{1}{2}mv^2. \tag{7.13}$$

Entering known values gives

$$\text{KE} = 0.5(30.0 \text{ kg})(0.500 \text{ m/s})^2, \tag{7.14}$$

which yields

$$\text{KE} = 3.75 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 3.75 \text{ J}. \tag{7.15}$$

Discussion

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Real World Connections: Center of Mass

Suppose we have two experimental carts, of equal mass, latched together on a track with a compressed spring between them. When the latch is released, the spring does 10 J of work on the carts (we'll see how in a couple of sections). The carts move relative to the spring, which is the center of mass of the system. However, the center of mass stays fixed. How can we consider the kinetic energy of this system?

By the work-energy theorem, the work done by the spring on the carts must turn into kinetic energy. So this system has 10 J of kinetic energy. The total kinetic energy of the system is the kinetic energy of the center of mass of the system relative to the fixed origin plus the kinetic energy of each cart relative to the center of mass. We know that the center of mass relative to the fixed origin does not move, and therefore all of the kinetic energy must be distributed among the carts relative to the center of mass. Since the carts have equal mass, they each receive an equal amount of kinetic energy, so each cart has 5.0 J of kinetic energy.

In our example, the forces between the spring and each cart are internal to the system. According to Newton's third law, these internal forces will cancel since they are equal and opposite in direction. However, this does not imply that these internal forces will not do work. Thus, the change in kinetic energy of the system is caused by work done by the force of the spring, and results in the motion of the two carts relative to the center of mass.

Example 7.3 Determining the Work to Accelerate a Package

Suppose that you push on the 30.0-kg package in **Figure 7.4** with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

Strategy and Concept for (a)

This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. (See **Figure 7.4**.) As expected, the net work is the net force times distance.

Solution for (a)

The net force is the push force minus friction, or $F_{\text{net}} = 120\text{ N} - 5.00\text{ N} = 115\text{ N}$. Thus the net work is

$$\begin{aligned} W_{\text{net}} &= F_{\text{net}}d = (115\text{ N})(0.800\text{ m}) \\ &= 92.0\text{ N}\cdot\text{m} = 92.0\text{ J}. \end{aligned} \tag{7.16}$$

Discussion for (a)

This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy. The net work equals the sum of the work done by each individual force.

Strategy and Concept for (b)

The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

Solution for (b)

The applied force does work.

$$\begin{aligned} W_{\text{app}} &= F_{\text{app}}d\cos(0^{\circ}) = F_{\text{app}}d \\ &= (120\text{ N})(0.800\text{ m}) \\ &= 96.0\text{ J} \end{aligned} \tag{7.17}$$

The friction force and displacement are in opposite directions, so that $\theta = 180^{\circ}$, and the work done by friction is

$$\begin{aligned} W_{\text{fr}} &= F_{\text{fr}}d\cos(180^{\circ}) = -F_{\text{fr}}d \\ &= -(5.00\text{ N})(0.800\text{ m}) \\ &= -4.00\text{ J}. \end{aligned} \tag{7.18}$$

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

$$\begin{aligned} W_{\text{gr}} &= 0, \\ W_{\text{N}} &= 0, \\ W_{\text{app}} &= 96.0\text{ J}, \\ W_{\text{fr}} &= -4.00\text{ J}. \end{aligned} \tag{7.19}$$

The total work done as the sum of the work done by each force is then seen to be

$$W_{\text{total}} = W_{\text{gr}} + W_{\text{N}} + W_{\text{app}} + W_{\text{fr}} = 92.0\text{ J}. \tag{7.20}$$

Discussion for (b)

The calculated total work W_{total} as the sum of the work by each force agrees, as expected, with the work W_{net} done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

Example 7.4 Determining Speed from Work and Energy

Find the speed of the package in **Figure 7.4** at the end of the push, using work and energy concepts.

Strategy

Here the work-energy theorem can be used, because we have just calculated the net work, W_{net} , and the initial kinetic energy, $\frac{1}{2}mv_0{}^2$. These calculations allow us to find the final kinetic energy, $\frac{1}{2}mv^2$, and thus the final speed v .

Solution

The work-energy theorem in equation form is

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^{\ 2}. \tag{7.21}$$

Solving for $\frac{1}{2}mv^2$ gives

$$\frac{1}{2}mv^2 = W_{\text{net}} + \frac{1}{2}mv_0^{\ 2}. \tag{7.22}$$

Thus,

$$\frac{1}{2}mv^2 = 92.0 \text{ J}+3.75 \text{ J} = 95.75 \text{ J}. \tag{7.23}$$

Solving for the final speed as requested and entering known values gives

$$\begin{aligned} v &= \sqrt{\frac{2(95.75 \text{ J})}{m}} = \sqrt{\frac{191.5 \text{ kg} \cdot \text{m}^2/\text{s}^2}{30.0 \text{ kg}}} \\ &= 2.53 \text{ m/s}. \end{aligned} \tag{7.24}$$

Discussion

Using work and energy, we not only arrive at an answer, we see that the final kinetic energy is the sum of the initial kinetic energy and the net work done on the package. This means that the work indeed adds to the energy of the package.

Example 7.5 Work and Energy Can Reveal Distance, Too

How far does the package in **Figure 7.4** coast after the push, assuming friction remains constant? Use work and energy considerations.

Strategy

We know that once the person stops pushing, friction will bring the package to rest. In terms of energy, friction does negative work until it has removed all of the package's kinetic energy. The work done by friction is the force of friction times the distance traveled times the cosine of the angle between the friction force and displacement; hence, this gives us a way of finding the distance traveled after the person stops pushing.

Solution

The normal force and force of gravity cancel in calculating the net force. The horizontal friction force is then the net force, and it acts opposite to the displacement, so $\theta = 180^\text{o}$. To reduce the kinetic energy of the package to zero, the work W_{fr} by friction must be minus the kinetic energy that the package started with plus what the package accumulated due to the pushing. Thus $W_{\text{fr}} = -95.75 \text{ J}$. Furthermore, $W_{\text{fr}} = fd' \cos \theta = -fd'$, where d' is the distance it takes to stop. Thus,

$$d' = -\frac{W_{\text{fr}}}{f} = -\frac{-95.75 \text{ J}}{5.00 \text{ N}}, \tag{7.25}$$

and so

$$d' = 19.2 \text{ m}. \tag{7.26}$$

Discussion

This is a reasonable distance for a package to coast on a relatively friction-free conveyor system. Note that the work done by friction is negative (the force is in the opposite direction of motion), so it removes the kinetic energy.

Some of the examples in this section can be solved without considering energy, but at the expense of missing out on gaining insights about what work and energy are doing in this situation. On the whole, solutions involving energy are generally shorter and easier than those using kinematics and dynamics alone.

7.3 Gravitational Potential Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain gravitational potential energy in terms of work done against gravity.
- Show that the gravitational potential energy of an object of mass *m* at height *h* on Earth is given by *PE*g = *mgh*.
- Show how knowledge of potential energy as a function of position can be used to simplify calculations and explain physical phenomena.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the calculation of component types of energy within the system whose sum is the total energy. **(S.P. 1.4, 2.1, 2.2)**
- **5.B.1.1** The student is able to set up a representation or model showing that a single object can only have kinetic energy and use information about that object to calculate its kinetic energy. **(S.P. 1.4, 2.2)**
- **5.B.1.2** The student is able to translate between a representation of a single object, which can only have kinetic energy, and a system that includes the object, which may have both kinetic and potential energies. **(S.P. 1.5)**

Work Done Against Gravity

Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass m through a height h, such as in **Figure 7.5**. If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight mg. The work done on the mass is then $W = Fd = mgh$. We define this to be the **gravitational potential energy** $(\mathrm{PE_g})$ put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the $\mathrm{PE_g}$ gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word “system”? Potential energy is a property of a system rather than of a single object—due to its physical position. An object’s gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth’s surface, but this point is arbitrary; what is important is the *difference* in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work equal to mgh on it, thereby increasing its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of $\mathrm{PE_g}$ to KE without explicitly considering the intermediate step of work. (See **Example 7.7**.) This shortcut makes it is easier to solve problems using energy (if possible) rather than explicitly using forces.

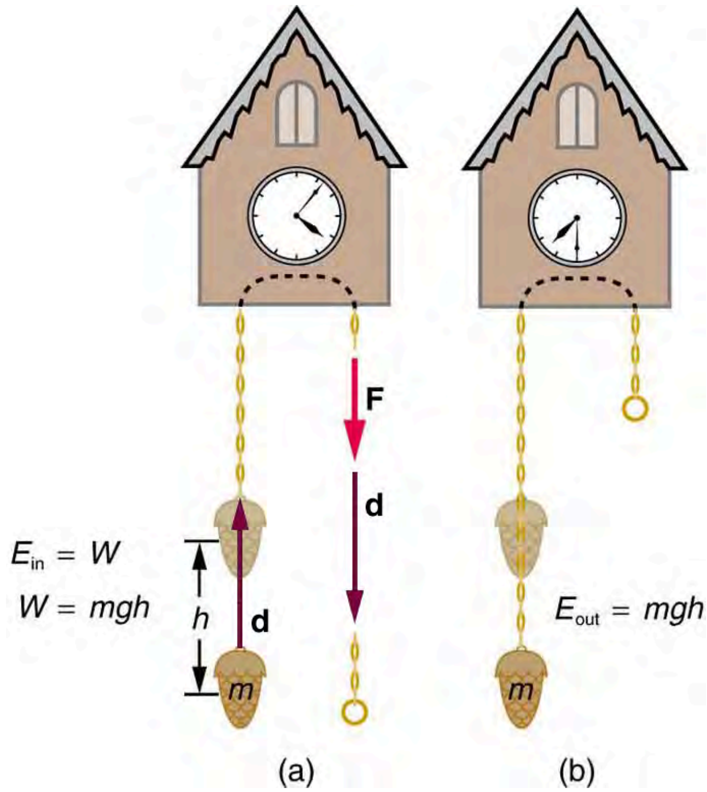

Figure 7.5 (a) The work done to lift the weight is stored in the mass-Earth system as gravitational potential energy. (b) As the weight moves downward, this gravitational potential energy is transferred to the cuckoo clock.

More precisely, we define the *change* in gravitational potential energy $\Delta \text{PE}_\text{g}$ to be

$$\Delta \text{PE}_\text{g} = mgh, \tag{7.27}$$

where, for simplicity, we denote the change in height by h rather than the usual Δh. Note that h is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

$$\begin{aligned} mgh &= (0.500\text{ kg})\left(9.80\text{ m/s}^2\right)(1.00\text{ m}) \\ &= 4.90\text{ kg}\cdot\text{m}^2/\text{s}^2 = 4.90\text{ J}. \end{aligned} \tag{7.28}$$

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, *without directly considering the force of gravity that does the work*.

Using Potential Energy to Simplify Calculations

The equation $\Delta \text{PE}_\text{g} = mgh$ applies for any path that has a change in height of h, not just when the mass is lifted straight up. (See **Figure 7.6**.) It is much easier to calculate mgh (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position h of a mass m is accompanied by a change in gravitational potential energy mgh, and we will avoid the equivalent but more difficult task of calculating work done by or against the gravitational force.

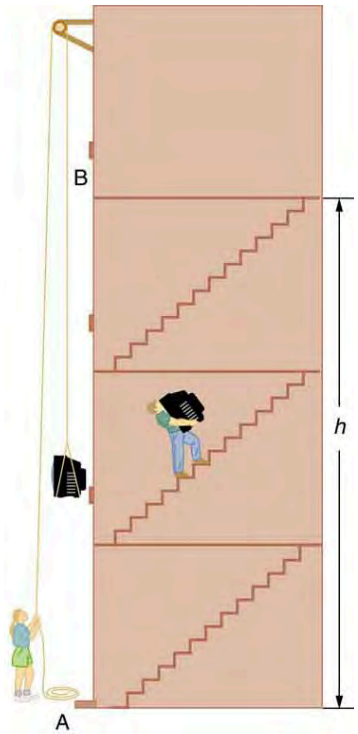

Figure 7.6 The change in gravitational potential energy $(\Delta \mathrm{PE_g})$ between points A and B is independent of the path. $\Delta \mathrm{PE_g} = mgh$ for any path between the two points. Gravity is one of a small class of forces where the work done by or against the force depends only on the starting and ending points, not on the path between them.

Example 7.6 The Force to Stop Falling

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

Strategy

This person's energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial $\mathrm{PE_g}$ is transformed into KE as he falls. The work done by the floor reduces this kinetic energy to zero.

Solution

The work done on the person by the floor as he stops is given by

$$W = Fd \cos \theta = -Fd, \tag{7.29}$$

with a minus sign because the displacement while stopping and the force from floor are in opposite directions $(\cos \theta = \cos 180^{\text{o}} = -1)$. The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height h:

$$\mathrm{KE} = -\Delta \mathrm{PE_g} = -mgh, \tag{7.30}$$

The distance d that the person's knees bend is much smaller than the height h of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work W done by the floor on the person stops the person and brings the person's kinetic energy to zero:

$$W = -\mathrm{KE} = mgh. \tag{7.31}$$

Combining this equation with the expression for W gives

$$-Fd = mgh. \tag{7.32}$$

Recalling that h is negative because the person fell *down*, the force on the knee joints is given by

$$F = -\frac{mgh}{d} = -\frac{(60.0\text{ kg})\left(9.80\text{ m/s}^2\right)(-3.00\text{ m})}{5.00\times10^{-3}\text{ m}} = 3.53\times10^{5}\text{ N}. \tag{7.33}$$

Discussion

Such a large force (500 times more than the person's weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo's hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump.(See Figure 7.7.)

Figure 7.7 The work done by the ground upon the kangaroo reduces its kinetic energy to zero as it lands. However, by applying the force of the ground on the hind legs over a longer distance, the impact on the bones is reduced. (credit: Chris Samuel, Flickr)

Example 7.7 Finding the Speed of a Roller Coaster from its Height

(a) What is the final speed of the roller coaster shown in Figure 7.8 if it starts from rest at the top of the 20.0 m hill and work done by frictional forces is negligible? (b) What is its final speed (again assuming negligible friction) if its initial speed is 5.00 m/s?

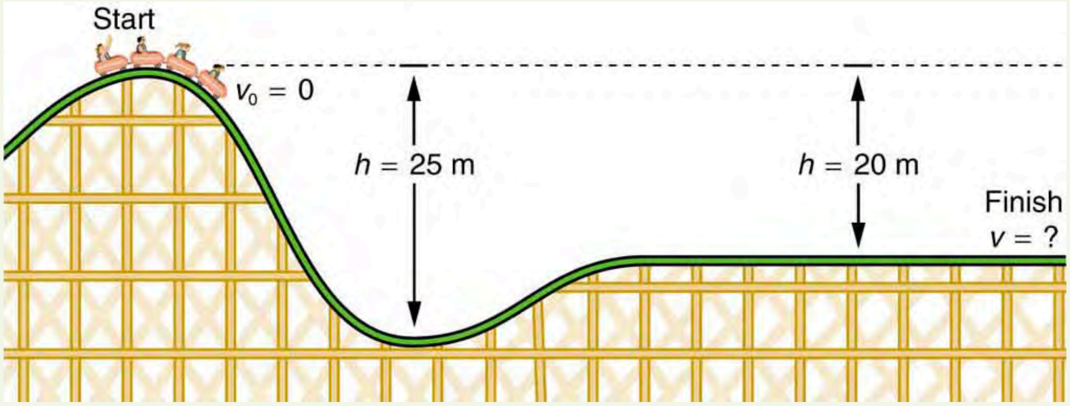

Figure 7.8 The speed of a roller coaster increases as gravity pulls it downhill and is greatest at its lowest point. Viewed in terms of energy, the roller-coaster-Earth system's gravitational potential energy is converted to kinetic energy. If work done by friction is negligible, all ΔPE_g is converted to KE.

Strategy

The roller coaster loses potential energy as it goes downhill. We neglect friction, so that the remaining force exerted by the track is the normal force, which is perpendicular to the direction of motion and does no work. The net work on the roller coaster is then done by gravity alone. The *loss* of gravitational potential energy from moving *downward* through a distance h equals the *gain* in kinetic energy. This can be written in equation form as $-\Delta\text{PE}_\text{g} = \Delta\text{KE}$. Using the equations for

$\mathrm{PE_g}$ and KE, we can solve for the final speed v, which is the desired quantity.

Solution for (a)

Here the initial kinetic energy is zero, so that $\Delta\mathrm{KE} = \frac{1}{2}mv^2$. The equation for change in potential energy states that $\Delta\mathrm{PE_g} = mgh$. Since h is negative in this case, we will rewrite this as $\Delta\mathrm{PE_g} = -mg\mid h\mid$ to show the minus sign clearly. Thus,

$$-\Delta\mathrm{PE_g} = \Delta\mathrm{KE} \tag{7.34}$$

becomes

$$mg\mid h\mid = \frac{1}{2}mv^2. \tag{7.35}$$

Solving for v, we find that mass cancels and that

$$v = \sqrt{2g\mid h\mid}. \tag{7.36}$$

Substituting known values,

$$\begin{aligned} v &= \sqrt{2\left(9.80\text{ m/s}^2\right)(20.0\text{ m})} \\ &= 19.8\text{ m/s}. \end{aligned} \tag{7.37}$$

Solution for (b)

Again $-\Delta\mathrm{PE_g} = \Delta\mathrm{KE}$. In this case there is initial kinetic energy, so $\Delta\mathrm{KE} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2$. Thus,

$$mg\mid h\mid = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2. \tag{7.38}$$

Rearranging gives

$$\frac{1}{2}mv^2 = mg\mid h\mid + \frac{1}{2}mv_0{}^2. \tag{7.39}$$

This means that the final kinetic energy is the sum of the initial kinetic energy and the gravitational potential energy. Mass again cancels, and

$$v = \sqrt{2g\mid h\mid + v_0{}^2}. \tag{7.40}$$

This equation is very similar to the kinematics equation $v = \sqrt{v_0{}^2 + 2ad}$, but it is more general—the kinematics equation is valid only for constant acceleration, whereas our equation above is valid for any path regardless of whether the object moves with a constant acceleration. Now, substituting known values gives

$$\begin{aligned} v &= \sqrt{2(9.80\text{ m/s}^2)(20.0\text{ m}) + (5.00\text{ m/s})^2} \\ &= 20.4\text{ m/s}. \end{aligned} \tag{7.41}$$

Discussion and Implications

First, note that mass cancels. This is quite consistent with observations made in **Falling Objects** that all objects fall at the same rate if friction is negligible. Second, only the speed of the roller coaster is considered; there is no information about its direction at any point. This reveals another general truth. When friction is negligible, the speed of a falling body depends only on its initial speed and height, and not on its mass or the path taken. For example, the roller coaster will have the same final speed whether it falls 20.0 m straight down or takes a more complicated path like the one in the figure. Third, and perhaps unexpectedly, the final speed in part (b) is greater than in part (a), but by far less than 5.00 m/s. Finally, note that speed can be found at *any* height along the way by simply using the appropriate value of h at the point of interest.

We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

Making Connections: Take-Home Investigation—Converting Potential to Kinetic Energy

One can study the conversion of gravitational potential energy into kinetic energy in this experiment. On a smooth, level surface, use a ruler of the kind that has a groove running along its length and a book to make an incline (see **Figure 7.9**).

Place a marble at the 10-cm position on the ruler and let it roll down the ruler. When it hits the level surface, measure the time it takes to roll one meter. Now place the marble at the 20-cm and the 30-cm positions and again measure the times it takes to roll 1 m on the level surface. Find the velocity of the marble on the level surface for all three positions. Plot velocity squared versus the distance traveled by the marble. What is the shape of each plot? If the shape is a straight line, the plot shows that the marble's kinetic energy at the bottom is proportional to its potential energy at the release point.

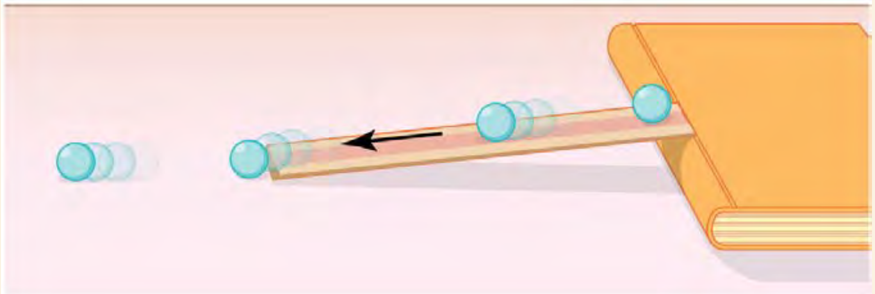

Figure 7.9 A marble rolls down a ruler, and its speed on the level surface is measured.

7.4 Conservative Forces and Potential Energy

Learning Objectives

By the end of this section, you will be able to:

- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces leads to conservation of mechanical energy.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the calculation of component types of energy within the system whose sum is the total energy. **(S.P. 1.4, 2.1, 2.2)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**
- **5.B.1.1** The student is able to set up a representation or model showing that a single object can only have kinetic energy and use information about that object to calculate its kinetic energy. **(S.P. 1.4, 2.2)**
- **5.B.1.2** The student is able to translate between a representation of a single object, which can only have kinetic energy, and a system that includes the object, which may have both kinetic and potential energies. **(S.P. 1.5)**
- **5.B.3.1** The student is able to describe and make qualitative and/or quantitative predictions about everyday examples of systems with internal potential energy. **(S.P. 2.2, 6.4, 7.2)**
- **5.B.3.2** The student is able to make quantitative calculations of the internal potential energy of a system from a description or diagram of that system. **(S.P. 1.4, 2.2)**
- **5.B.3.3** The student is able to apply mathematical reasoning to create a description of the internal potential energy of a system from a description or diagram of the objects and interactions in that system. **(S.P. 1.4, 2.2)**

Potential Energy and Conservative Forces

Work is done by a force, and some forces, such as weight, have special characteristics. A **conservative force** is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a **potential energy** (PE) for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is *conservative*. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

Potential Energy and Conservative Forces

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

A conservative force is one for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken.

We can define a potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration depends on the configuration, not the path followed, and is the potential energy added.

Real World Connections: Energy of a Bowling Ball

How much energy does a bowling ball have? (Just think about it for a minute.)

If you are thinking that you need more information, you're right. If we can measure the ball's velocity, then determining its kinetic energy is simple. Note that this does require defining a reference frame in which to measure the velocity. Determining the ball's potential energy also requires more information. You need to know its height above the ground, which requires a reference frame of the ground. Without the ground—in other words, Earth—the ball does not classically have potential energy. Potential energy comes from the interaction between the ball and the ground. Another way of thinking about this is to compare the ball's potential energy on Earth and on the Moon. A bowling ball a certain height above Earth is going to have more potential energy than the same bowling ball the same height above the surface of the Moon, because Earth has greater mass than the Moon and therefore exerts more gravity on the ball. Thus, potential energy requires a system of at least two objects, or an object with an internal structure of at least two parts.

Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring ($\mathrm{PE_s}$). We calculate the work done to stretch or compress a spring that obeys Hooke's law. (Hooke's law was examined in **Elasticity: Stress and Strain**, and states that the magnitude of force F on the spring and the resulting deformation ΔL are proportional, $F = k\Delta L$.) (See **Figure 7.10**.) For our spring, we will replace ΔL (the amount of deformation produced by a force F) by the distance x that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude $F = kx$, where k is the spring's force constant. The force increases linearly from 0 at the start to kx in the fully stretched position. The average force is $kx/2$. Thus the work done in stretching or compressing the spring is $W_\mathrm{s} = Fd = \left(\frac{kx}{2}\right)x = \frac{1}{2}kx^2$. Alternatively, we noted in **Kinetic Energy and the Work-Energy Theorem** that the area under a graph of F vs. x is the work done by the force. In **Figure 7.10**(c) we see that this area is also $\frac{1}{2}kx^2$. We therefore define the **potential energy of a spring**, $\mathrm{PE_s}$, to be

$$\mathrm{PE_s} = \frac{1}{2}kx^2, \tag{7.42}$$

where k is the spring's force constant and x is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance x . The potential energy of the spring $\mathrm{PE_s}$ does not depend on the path taken; it depends only on the stretch or squeeze x in the final configuration.

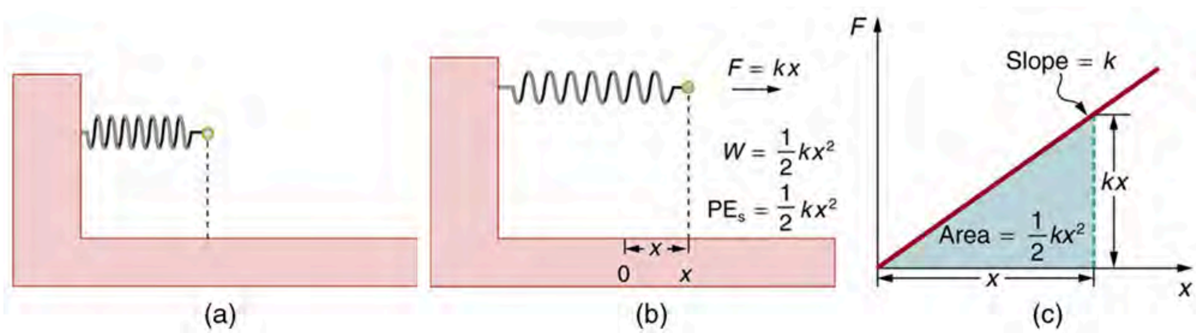

Figure 7.10 (a) An undeformed spring has no $\mathrm{PE_s}$ stored in it. (b) The force needed to stretch (or compress) the spring a distance x has a magnitude $F = kx$, and the work done to stretch (or compress) it is $\frac{1}{2}kx^2$. Because the force is conservative, this work is stored as potential energy $(\mathrm{PE_s})$ in the spring, and it can be fully recovered. (c) A graph of F vs. x has a slope of k , and the area under the graph is $\frac{1}{2}kx^2$. Thus the work done or potential energy stored is $\frac{1}{2}kx^2$.

The equation $\mathrm{PE_s} = \frac{1}{2}kx^2$ has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of **potential energy** is energy due to position, shape, or configuration. For shape or position deformations, stored energy is $\mathrm{PE_s} = \frac{1}{2}kx^2$, where k is the force constant of the particular system and x is its deformation. Another example is seen in **Figure 7.11** for a guitar string.

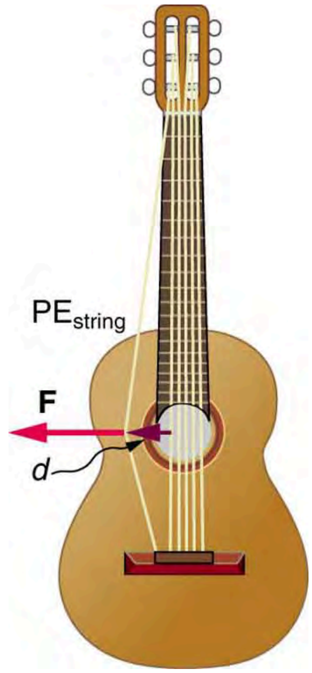

Figure 7.11 Work is done to deform the guitar string, giving it potential energy. When released, the potential energy is converted to kinetic energy and back to potential as the string oscillates back and forth. A very small fraction is dissipated as sound energy, slowly removing energy from the string.

Conservation of Mechanical Energy

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2 = \Delta\mathrm{KE}. \tag{7.43}$$

If only conservative forces act, then

$$W_{\text{net}} = W_{\text{c}}, \tag{7.44}$$

where W_{c} is the total work done by all conservative forces. Thus,

$$W_{\text{c}} = \Delta\mathrm{KE}. \tag{7.45}$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is, $W_{\text{c}} = -\Delta\mathrm{PE}$. Therefore,

$$-\Delta\mathrm{PE} = \Delta\mathrm{KE} \tag{7.46}$$

or

$$\Delta\mathrm{KE} + \Delta\mathrm{PE} = 0. \tag{7.47}$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

$$\left.\begin{array}{rc} & \mathrm{KE} + \mathrm{PE} = \text{constant} \\ \text{or} & \\ & \mathrm{KE_i} + \mathrm{PE_i} = \mathrm{KE_f} + \mathrm{PE_f} \end{array}\right\}\text{(conservative forces only)}, \tag{7.48}$$

where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the **conservation of mechanical energy** principle. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its **mechanical energy**, $(\mathrm{KE} + \mathrm{PE})$. In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE , with the total energy remaining constant.

The internal energy of a system is the sum of the kinetic energies of all of its elements, plus the potential energy due to all of the interactions due to conservative forces between all of the elements.

Real World Connections

Consider a wind-up toy, such as a car. It uses a spring system to store energy. The amount of energy stored depends only on how many times it is wound, not how quickly or slowly the winding happens. Similarly, a dart gun using compressed air stores energy in its internal structure. In this case, the energy stored inside depends only on how many times it is pumped, not how quickly or slowly the pumping is done. The total energy put into the system, whether through winding or pumping, is equal to the total energy conserved in the system (minus any energy loss in the system due to interactions between its parts, such as air leaks in the dart gun). Since the internal energy of the system is conserved, you can calculate the amount of stored energy by measuring the kinetic energy of the system (the moving car or dart) when the potential energy is released.

Example 7.8 Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car

A 0.100-kg toy car is propelled by a compressed spring, as shown in Figure 7.12. The car follows a track that rises 0.180 m above the starting point. The spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car is going before it starts up the slope and (b) how fast it is going at the top of the slope.

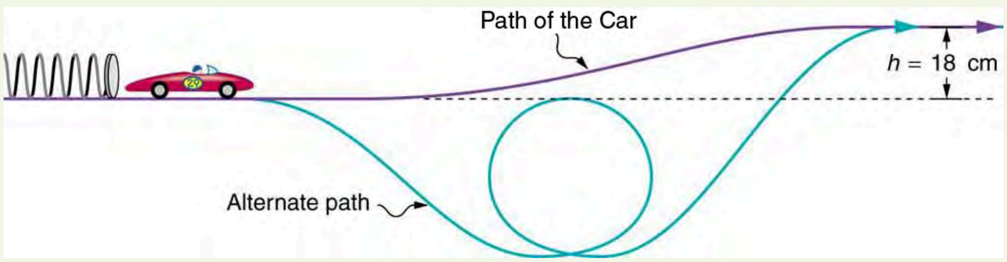

Figure 7.12 A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.

Strategy

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

$$\text{KE}_\text{i} + \text{PE}_\text{i} = \text{KE}_\text{f} + \text{PE}_\text{f} \tag{7.49}$$

or

$$\frac{1}{2}mv_\text{i}^2 + mgh_\text{i} + \frac{1}{2}kx_\text{i}^2 = \frac{1}{2}mv_\text{f}^2 + mgh_\text{f} + \frac{1}{2}kx_\text{f}^2, \tag{7.50}$$

where h is the height (vertical position) and x is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

Solution for (a)

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both h_i and h_f are zero. Furthermore, the initial speed v_i is zero and the final compression of the spring x_f is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

$$\frac{1}{2}kx_\text{i}^2 = \frac{1}{2}mv_\text{f}^2. \tag{7.51}$$

In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

$$\begin{aligned} v_\text{f} &= \sqrt{\frac{k}{m}}x_\text{i} \\ &= \sqrt{\frac{250.0\text{ N/m}}{0.100\text{ kg}}}(0.0400\text{ m}) \\ &= 2.00\text{ m/s}. \end{aligned} \tag{7.52}$$

Solution for (b)

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between. Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

$$\frac{1}{2}kx_{\mathrm{i}}^2 = \frac{1}{2}mv_{\mathrm{f}}^2 + mgh_{\mathrm{f}}. \tag{7.53}$$

This form of the equation means that the spring's initial potential energy is converted partly to gravitational potential energy and partly to kinetic energy. The final speed at the top of the slope will be less than at the bottom. Solving for v_{f} and substituting known values gives

$$\begin{aligned} v_{\mathrm{f}} &= \sqrt{\frac{kx_{\mathrm{i}}^2}{m} - 2gh_{\mathrm{f}}} \\ &= \sqrt{\left(\frac{250.0\ \mathrm{N/m}}{0.100\ \mathrm{kg}}\right)(0.0400\ \mathrm{m})^2 - 2(9.80\ \mathrm{m/s}^2)(0.180\ \mathrm{m})} \\ &= 0.687\ \mathrm{m/s}. \end{aligned} \tag{7.54}$$

Discussion

Another way to solve this problem is to realize that the car's kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Applying the Science Practices: Potential Energy in a Spring

Suppose you are running an experiment in which two 250 g carts connected by a spring (with spring constant 120 N/m) are run into a solid block, and the compression of the spring is measured. In one run of this experiment, the spring was measured to compress from its rest length of 5.0 cm to a minimum length of 2.0 cm. What was the potential energy stored in this system?

Answer

Note that the change in length of the spring is 3.0 cm. Hence we can apply Equation 7.42 to find that the potential energy is PE = (1/2)(120 N/m)(0.030 m)^2 = 0.0541 J.

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in **Example 7.8**. Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

PhET Explorations: Energy Skate Park

Learn about conservation of energy with a skater dude! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

PhET Interactive Simulation

Figure 7.13 Energy Skate Park (http://cnx.org/content/m55076/1.5/energy-skate-park_en.jar)

7.5 Nonconservative Forces

Learning Objectives

By the end of this section, you will be able to:

- Define nonconservative forces and explain how they affect mechanical energy.
- Show how the principle of conservation of energy can be applied by treating the conservative forces in terms of their potential energies and any nonconservative forces in terms of the work they do.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.2** The student is able to predict changes in the total energy of a system due to changes in position and speed of objects or frictional interactions within the system. **(S.P. 6.4)**

- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**

Nonconservative Forces and Friction

Forces are either conservative or nonconservative. Conservative forces were discussed in **Conservative Forces and Potential Energy**. A **nonconservative force** is one for which work depends on the path taken. Friction is a good example of a nonconservative force. As illustrated in **Figure 7.14**, work done against friction depends on the length of the path between the starting and ending points. Because of this dependence on path, there is no potential energy associated with nonconservative forces. An important characteristic is that the work done by a nonconservative force *adds or removes mechanical energy from a system*. **Friction**, for example, creates **thermal energy** that dissipates, removing energy from the system. Furthermore, even if the thermal energy is retained or captured, it cannot be fully converted back to work, so it is lost or not recoverable in that sense as well.

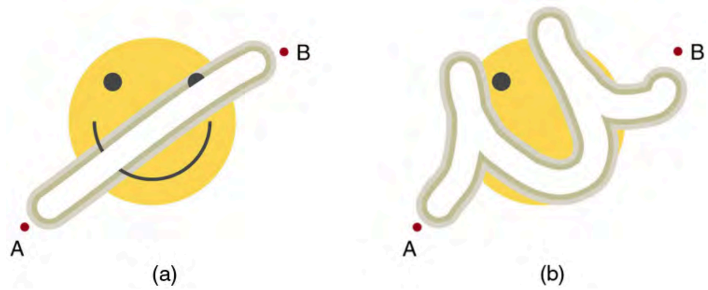

Figure 7.14 The amount of the happy face erased depends on the path taken by the eraser between points A and B, as does the work done against friction. Less work is done and less of the face is erased for the path in (a) than for the path in (b). The force here is friction, and most of the work goes into thermal energy that subsequently leaves the system (the happy face plus the eraser). The energy expended cannot be fully recovered.

How Nonconservative Forces Affect Mechanical Energy

Mechanical energy *may* not be conserved when nonconservative forces act. For example, when a car is brought to a stop by friction on level ground, it loses kinetic energy, which is dissipated as thermal energy, reducing its mechanical energy. **Figure 7.15** compares the effects of conservative and nonconservative forces. We often choose to understand simpler systems such as that described in **Figure 7.15**(a) first before studying more complicated systems as in **Figure 7.15**(b).

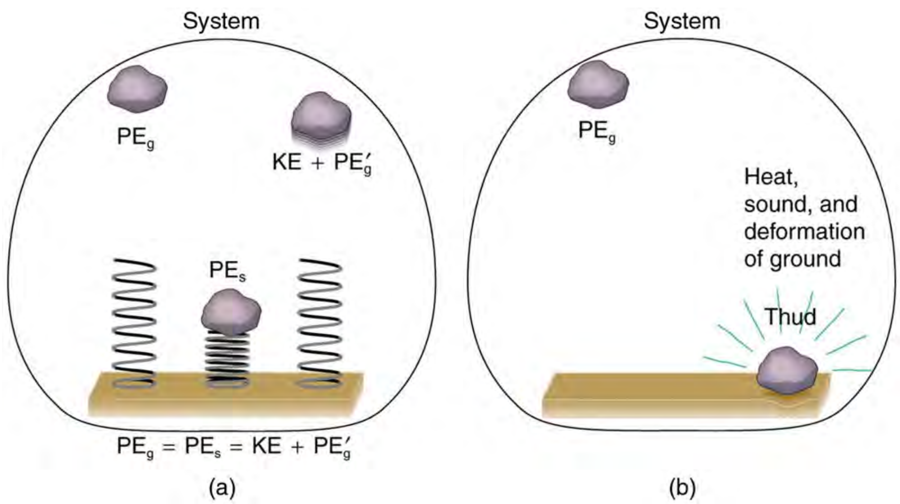

Figure 7.15 Comparison of the effects of conservative and nonconservative forces on the mechanical energy of a system. (a) A system with only conservative forces. When a rock is dropped onto a spring, its mechanical energy remains constant (neglecting air resistance) because the force in the spring is conservative. The spring can propel the rock back to its original height, where it once again has only potential energy due to gravity. (b) A system with nonconservative forces. When the same rock is dropped onto the ground, it is stopped by nonconservative forces that dissipate its mechanical energy as thermal energy, sound, and surface distortion. The rock has lost mechanical energy.

How the Work-Energy Theorem Applies

Now let us consider what form the work-energy theorem takes when both conservative and nonconservative forces act. We will see that the work done by nonconservative forces equals the change in the mechanical energy of a system. As noted in **Kinetic Energy and the Work-Energy Theorem**, the work-energy theorem states that the net work on a system equals the change in its kinetic energy, or $W_{\text{net}} = \Delta\text{KE}$. The net work is the sum of the work by nonconservative forces plus the work by conservative forces. That is,

$$W_{\text{net}} = W_{\text{nc}} + W_{\text{c}}, \tag{7.55}$$

so that

$$W_{\text{nc}} + W_{\text{c}} = \Delta\text{KE}, \tag{7.56}$$

where W_{nc} is the total work done by all nonconservative forces and W_{c} is the total work done by all conservative forces.

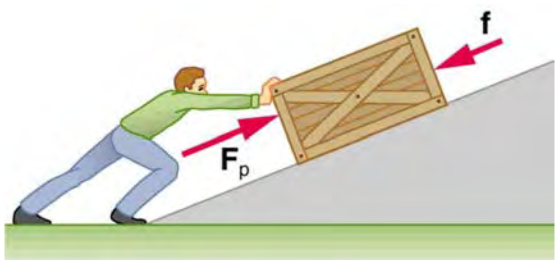

Figure 7.16 A person pushes a crate up a ramp, doing work on the crate. Friction and gravitational force (not shown) also do work on the crate; both forces oppose the person's push. As the crate is pushed up the ramp, it gains mechanical energy, implying that the work done by the person is greater than the work done by friction.

Consider **Figure 7.16**, in which a person pushes a crate up a ramp and is opposed by friction. As in the previous section, we note that work done by a conservative force comes from a loss of gravitational potential energy, so that $W_{\text{c}} = -\Delta\text{PE}$. Substituting this equation into the previous one and solving for W_{nc} gives

$$W_{\text{nc}} = \Delta\text{KE} + \Delta\text{PE}. \tag{7.57}$$

This equation means that the total mechanical energy $(\text{KE} + \text{PE})$ changes by exactly the amount of work done by nonconservative forces. In **Figure 7.16**, this is the work done by the person minus the work done by friction. So even if energy is not conserved for the system of interest (such as the crate), we know that an equal amount of work was done to cause the change in total mechanical energy.

We rearrange $W_{\text{nc}} = \Delta\text{KE} + \Delta\text{PE}$ to obtain

$$\text{KE}_{\text{i}} + \text{PE}_{\text{i}} + W_{\text{nc}} = \text{KE}_{\text{f}} + \text{PE}_{\text{f}}. \tag{7.58}$$

This means that the amount of work done by nonconservative forces adds to the mechanical energy of a system. If W_{nc} is positive, then mechanical energy is increased, such as when the person pushes the crate up the ramp in **Figure 7.16**. If W_{nc} is negative, then mechanical energy is decreased, such as when the rock hits the ground in **Figure 7.15**(b). If W_{nc} is zero, then mechanical energy is conserved, and nonconservative forces are balanced. For example, when you push a lawn mower at constant speed on level ground, your work done is removed by the work of friction, and the mower has a constant energy.

Applying Energy Conservation with Nonconservative Forces

When no change in potential energy occurs, applying $\text{KE}_{\text{i}} + \text{PE}_{\text{i}} + W_{\text{nc}} = \text{KE}_{\text{f}} + \text{PE}_{\text{f}}$ amounts to applying the work-energy theorem by setting the change in kinetic energy to be equal to the net work done on the system, which in the most general case includes both conservative and nonconservative forces. But when seeking instead to find a change in total mechanical energy in situations that involve changes in both potential and kinetic energy, the previous equation $\text{KE}_{\text{i}} + \text{PE}_{\text{i}} + W_{\text{nc}} = \text{KE}_{\text{f}} + \text{PE}_{\text{f}}$ says that you can start by finding the change in mechanical energy that would have resulted from just the conservative forces, including the potential energy changes, and add to it the work done, with the proper sign, by any nonconservative forces involved.

Example 7.9 Calculating Distance Traveled: How Far a Baseball Player Slides

Consider the situation shown in **Figure 7.17**, where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a constant 450 N.

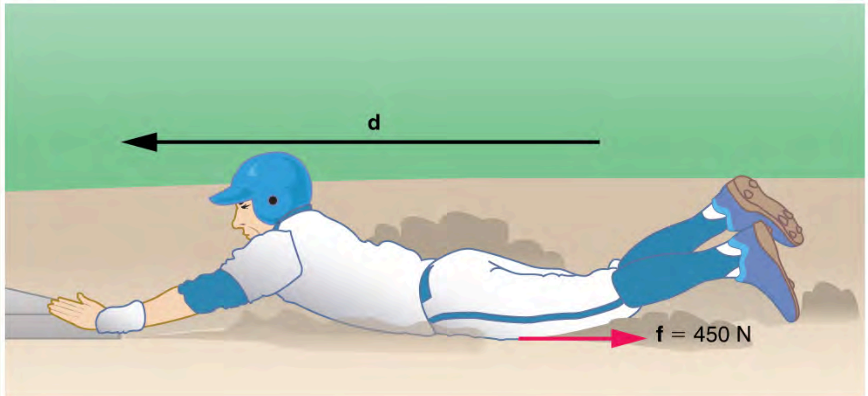

Figure 7.17 The baseball player slides to a stop in a distance d. In the process, friction removes the player's kinetic energy by doing an amount of work fd equal to the initial kinetic energy.

Strategy

Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction, which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because $\mathbf{f}$ is in the opposite direction of the motion (that is, $\theta = 180^\circ$, and so $\cos\theta = -1$). Thus $W_{\text{nc}} = -fd$. The equation simplifies to

$$\frac{1}{2}m{v_{\text{i}}}^2 - fd = 0 \tag{7.59}$$

or

$$fd = \frac{1}{2}m{v_{\text{i}}}^2. \tag{7.60}$$

This equation can now be solved for the distance d.

Solution

Solving the previous equation for d and substituting known values yields

$$\begin{aligned} d &= \frac{m{v_{\text{i}}}^2}{2f} \\ &= \frac{(65.0\text{ kg})(6.00\text{ m/s})^2}{(2)(450\text{ N})} \\ &= 2.60\text{ m.} \end{aligned} \tag{7.61}$$

Discussion

The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy. For example, you must work harder to stop a truck, with its large mechanical energy, than to stop a mosquito.

Example 7.10 Calculating Distance Traveled: Sliding Up an Incline

Suppose that the player from **Example 7.9** is running up a hill having a 5.00° incline upward with a surface similar to that in the baseball stadium. The player slides with the same initial speed, and the frictional force is still 450 N. Determine how far he slides.

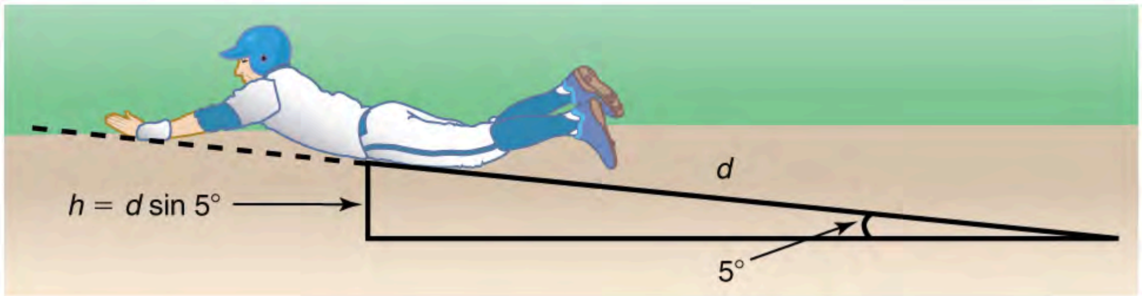

Figure 7.18 The same baseball player slides to a stop on a 5.00° slope.

Strategy

In this case, the work done by the nonconservative friction force on the player reduces the mechanical energy he has from his kinetic energy at zero height, to the final mechanical energy he has by moving through distance d to reach height h along the hill, with $h = d \sin 5.00^\circ$. This is expressed by the equation

$$\mathrm{KE} + \mathrm{PE_i} + W_{\mathrm{nc}} = \mathrm{KE_f} + \mathrm{PE_f}. \tag{7.62}$$

Solution

The work done by friction is again $W_{\mathrm{nc}} = -fd$; initially the potential energy is $\mathrm{PE_i} = mg \cdot 0 = 0$ and the kinetic energy is $\mathrm{KE_i} = \frac{1}{2}mv_{\mathrm{i}}^{\ 2}$; the final energy contributions are $\mathrm{KE_f} = 0$ for the kinetic energy and $\mathrm{PE_f} = mgh = mgd \sin\theta$ for the potential energy.

Substituting these values gives

$$\frac{1}{2}mv_{\mathrm{i}}^{\ 2} + 0 + \left(-fd\right) = 0 + mgd \sin\theta. \tag{7.63}$$

Solve this for d to obtain

$$\begin{aligned} d &= \frac{\left(\frac{1}{2}\right)mv_{\mathrm{i}}^{\ 2}}{f + mg \sin\theta} \\ &= \frac{(0.5)(65.0\ \mathrm{kg})(6.00\ \mathrm{m/s})^2}{450\ \mathrm{N}{+}(65.0\ \mathrm{kg})(9.80\ \mathrm{m/s^2})\ \sin(5.00^\circ)} \\ &= 2.31\ \mathrm{m}. \end{aligned} \tag{7.64}$$

Discussion

As might have been expected, the player slides a shorter distance by sliding uphill. Note that the problem could also have been solved in terms of the forces directly and the work energy theorem, instead of using the potential energy. This method would have required combining the normal force and force of gravity vectors, which no longer cancel each other because they point in different directions, and friction, to find the net force. You could then use the net force and the net work to find the distance d that reduces the kinetic energy to zero. By applying conservation of energy and using the potential energy instead, we need only consider the gravitational potential energy mgh, without combining and resolving force vectors. This simplifies the solution considerably.

Making Connections: Take-Home Investigation—Determining Friction from the Stopping Distance

This experiment involves the conversion of gravitational potential energy into thermal energy. Use the ruler, book, and marble from **Making Connections: Take-Home Investigation—Converting Potential to Kinetic Energy**. In addition, you will need a foam cup with a small hole in the side, as shown in **Figure 7.19**. From the 10-cm position on the ruler, let the marble roll into the cup positioned at the bottom of the ruler. Measure the distance d the cup moves before stopping. What forces caused it to stop? What happened to the kinetic energy of the marble at the bottom of the ruler? Next, place the marble at the 20-cm and the 30-cm positions and again measure the distance the cup moves after the marble enters it. Plot the distance the cup moves versus the initial marble position on the ruler. Is this relationship linear?

With some simple assumptions, you can use these data to find the coefficient of kinetic friction μ_{k} of the cup on the table. The force of friction f on the cup is $\mu_{\mathrm{k}} N$, where the normal force N is just the weight of the cup plus the marble. The normal force and force of gravity do no work because they are perpendicular to the displacement of the cup, which moves horizontally. The work done by friction is fd . You will need the mass of the marble as well to calculate its initial kinetic energy.

It is interesting to do the above experiment also with a steel marble (or ball bearing). Releasing it from the same positions on the ruler as you did with the glass marble, is the velocity of this steel marble the same as the velocity of the marble at the bottom of the ruler? Is the distance the cup moves proportional to the mass of the steel and glass marbles?

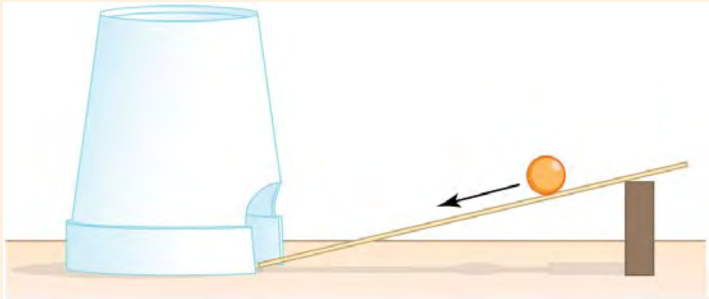

Figure 7.19 Rolling a marble down a ruler into a foam cup.

PhET Explorations: The Ramp

Explore forces, energy and work as you push household objects up and down a ramp. Lower and raise the ramp to see how the angle of inclination affects the parallel forces acting on the file cabinet. Graphs show forces, energy and work.

PhET Interactive Simulation

Figure 7.20 The Ramp (http://cnx.org/content/m55047/1.5/the-ramp_en.jar)

7.6 Conservation of Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain the law of the conservation of energy.
- Describe some of the many forms of energy.
- Define efficiency of an energy conversion process as the fraction left as useful energy or work, rather than being transformed, for example, into thermal energy.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.2** The student is able to predict changes in the total energy of a system due to changes in position and speed of objects or frictional interactions within the system. **(S.P. 6.4)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**
- **4.C.2.2** The student is able to apply the concepts of conservation of energy and the work-energy theorem to determine qualitatively and/or quantitatively that work done on a two-object system in linear motion will change the kinetic energy of the center of mass of the system, the potential energy of the systems, and/or the internal energy of the system. **(S.P. 1.4, 2.2, 7.2)**
- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. **(S.P. 6.4, 7.2)**
- **5.B.5.4** The student is able to make claims about the interaction between a system and its environment in which the environment exerts a force on the system, thus doing work on the system and changing the energy of the system (kinetic energy plus potential energy). **(S.P. 6.4, 7.2)**
- **5.B.5.5** The student is able to predict and calculate the energy transfer to (i.e., the work done on) an object or system from information about a force exerted on the object or system through a distance. **(S.P. 2.2, 6.4)**

Law of Conservation of Energy

Energy, as we have noted, is conserved, making it one of the most important physical quantities in nature. The **law of**

conservation of energy can be stated as follows:

Total energy is constant in any process. It may change in form or be transferred from one system to another, but the total remains the same.

We have explored some forms of energy and some ways it can be transferred from one system to another. This exploration led to the definition of two major types of energy—mechanical energy $(\mathrm{KE} + \mathrm{PE})$ and energy transferred via work done by nonconservative forces (W_{nc}). But energy takes *many* other forms, manifesting itself in *many* different ways, and we need to be able to deal with all of these before we can write an equation for the above general statement of the conservation of energy.

Other Forms of Energy than Mechanical Energy

At this point, we deal with all other forms of energy by lumping them into a single group called other energy (OE). Then we can state the conservation of energy in equation form as

$$\mathrm{KE_i} + \mathrm{PE_i} + W_{\mathrm{nc}} + \mathrm{OE_i} = \mathrm{KE_f} + \mathrm{PE_f} + \mathrm{OE_f}. \tag{7.65}$$

All types of energy and work can be included in this very general statement of conservation of energy. Kinetic energy is KE, work done by a conservative force is represented by PE, work done by nonconservative forces is W_{nc}, and all other energies are included as OE. This equation applies to all previous examples; in those situations OE was constant, and so it subtracted out and was not directly considered.

Making Connections: Usefulness of the Energy Conservation Principle

The fact that energy is conserved and has many forms makes it very important. You will find that energy is discussed in many contexts, because it is involved in all processes. It will also become apparent that many situations are best understood in terms of energy and that problems are often most easily conceptualized and solved by considering energy.

When does OE play a role? One example occurs when a person eats. Food is oxidized with the release of carbon dioxide, water, and energy. Some of this chemical energy is converted to kinetic energy when the person moves, to potential energy when the person changes altitude, and to thermal energy (another form of OE).

Some of the Many Forms of Energy

What are some other forms of energy? You can probably name a number of forms of energy not yet discussed. Many of these will be covered in later chapters, but let us detail a few here. **Electrical energy** is a common form that is converted to many other forms and does work in a wide range of practical situations. Fuels, such as gasoline and food, carry **chemical energy** that can be transferred to a system through oxidation. Chemical fuel can also produce electrical energy, such as in batteries. Batteries can in turn produce light, which is a very pure form of energy. Most energy sources on Earth are in fact stored energy from the energy we receive from the Sun. We sometimes refer to this as **radiant energy**, or electromagnetic radiation, which includes visible light, infrared, and ultraviolet radiation. **Nuclear energy** comes from processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into the energy of sunlight, into electrical energy in power plants, and into the energy of the heat transfer and blast in weapons. Atoms and molecules inside all objects are in random motion. This internal mechanical energy from the random motions is called **thermal energy**, because it is related to the temperature of the object. These and all other forms of energy can be converted into one another and can do work.

Real World Connections: Open or Closed System?

Consider whether the following systems are open or closed: a car, a spring-operated dart gun, and the system shown in **Figure 7.15**(a).

A car is not a closed system. You add energy in the form of more gas in the tank (or charging the batteries), and energy is lost due to air resistance and friction.

A spring-operated dart gun is not a closed system. You have to initially compress the spring. Once that has been done, however, the dart gun and dart can be treated as a closed system. All of the energy remains in the system consisting of these two objects.

Figure 7.15(a) is an example of a closed system, once it has been started. All of the energy in the system remains there; none is brought in from outside or leaves.

Table 7.1 gives the amount of energy stored, used, or released from various objects and in various phenomena. The range of energies and the variety of types and situations is impressive.

Problem-Solving Strategies for Energy

You will find the following problem-solving strategies useful whenever you deal with energy. The strategies help in organizing and reinforcing energy concepts. In fact, they are used in the examples presented in this chapter. The familiar general problem-solving strategies presented earlier—involving identifying physical principles, knowns, and unknowns, checking units, and so on—continue to be relevant here.

Step 1. Determine the system of interest and identify what information is given and what quantity is to be calculated. A sketch will help.

Step 2. Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step 3 or step 4.

Step 3. If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is

$$\mathrm{KE_i} + \mathrm{PE_i} = \mathrm{KE_f} + \mathrm{PE_f}. \tag{7.66}$$

Step 4. If you know the potential energy for only some of the forces, possibly because some of them are nonconservative and do not have a potential energy, or if there are other energies that are not easily treated in terms of force and work, then the conservation of energy law in its most general form must be used.

$$\mathrm{KE_i} + \mathrm{PE_i} + W_{\mathrm{nc}} + \mathrm{OE_i} = \mathrm{KE_f} + \mathrm{PE_f} + \mathrm{OE_f}. \tag{7.67}$$

In most problems, one or more of the terms is zero, simplifying its solution. Do not calculate W_{c} , the work done by conservative forces; it is already incorporated in the PE terms.

Step 5. You have already identified the types of work and energy involved (in step 2). Before solving for the unknown, *eliminate terms wherever possible* to simplify the algebra. For example, choose $h = 0$ at either the initial or final point, so that $\mathrm{PE_g}$ is zero there. Then solve for the unknown in the customary manner.

Step 6. *Check the answer to see if it is reasonable*. Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on. Also check to see that the numerical value obtained is reasonable. For example, the final speed of a skateboarder who coasts down a 3-m-high ramp could reasonably be 20 km/h, but *not* 80 km/h.

Transformation of Energy

The transformation of energy from one form into others is happening all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. In a larger example, the chemical energy contained in coal is converted into thermal energy as it burns to turn water into steam in a boiler. This thermal energy in the steam in turn is converted to mechanical energy as it spins a turbine, which is connected to a generator to produce electrical energy. (In all of these examples, not all of the initial energy is converted into the forms mentioned. This important point is discussed later in this section.)

Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell (see **Figure 7.21**) produces electricity, which in turn can be used to run an electric motor. Energy is converted from the primary source of solar energy into electrical energy and then into mechanical energy.

Figure 7.21 Solar energy is converted into electrical energy by solar cells, which is used to run a motor in this solar-power aircraft. (credit: NASA)

Table 7.1 Energy of Various Objects and Phenomena

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Energy released in a supernova	10^{44}
Fusion of all the hydrogen in Earth's oceans	10^{34}
Annual world energy use	4×10^{20}
Large fusion bomb (9 megaton)	3.8×10^{16}
1 kg hydrogen (fusion to helium)	6.4×10^{14}
1 kg uranium (nuclear fission)	8.0×10^{13}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}
90,000-ton aircraft carrier at 30 knots	1.1×10^{10}
1 barrel crude oil	5.9×10^{9}
1 ton TNT	4.2×10^{9}
1 gallon of gasoline	1.2×10^{8}
Daily home electricity use (developed countries)	7×10^{7}
Daily adult food intake (recommended)	1.2×10^{7}
1000-kg car at 90 km/h	3.1×10^{5}
1 g fat (9.3 kcal)	3.9×10^{4}
ATP hydrolysis reaction	3.2×10^{4}
1 g carbohydrate (4.1 kcal)	1.7×10^{4}
1 g protein (4.1 kcal)	1.7×10^{4}
Tennis ball at 100 km/h	22
Mosquito $\left(10^{-2}\text{ g at }0.5\text{ m/s}\right)$	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

Efficiency

Even though energy is conserved in an energy conversion process, the output of *useful energy* or work will be less than the energy input. The **efficiency** Eff of an energy conversion process is defined as

$$\text{Efficien}\quad (Eff) = \frac{\text{useful energy or work output}}{\text{total energy input}} = \frac{W_{\text{out}}}{E_{\text{in}}}. \tag{7.68}$$

Table 7.2 lists some efficiencies of mechanical devices and human activities. In a coal-fired power plant, for example, about 40% of the chemical energy in the coal becomes useful electrical energy. The other 60% transforms into other (perhaps less useful) energy forms, such as thermal energy, which is then released to the environment through combustion gases and cooling towers.

Table 7.2 Efficiency of the Human Body and Mechanical Devices

Activity/device	Efficiency (%)[1]
Cycling and climbing	20
Swimming, surface	2
Swimming, submerged	4
Shoveling	3
Weightlifting	9
Steam engine	17
Gasoline engine	30
Diesel engine	35
Nuclear power plant	35
Coal power plant	42
Electric motor	98
Compact fluorescent light	20
Gas heater (residential)	90
Solar cell	10

PhET Explorations: Masses and Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energies for each spring.

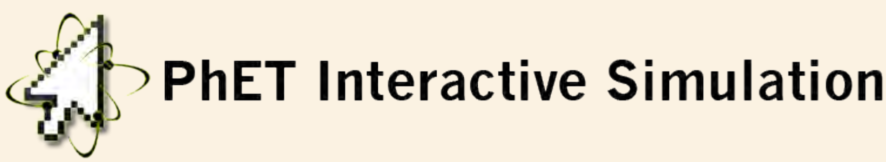

Figure 7.22 Masses and Springs (http://cnx.org/content/m55049/1.4/mass-spring-lab_en.jar)

7.7 Power

Learning Objectives

By the end of this section, you will be able to:

- Calculate power by calculating changes in energy over time.
- Examine power consumption and calculations of the cost of energy consumed.

What is Power?

Power—the word conjures up many images: a professional football player muscling aside his opponent, a dragster roaring away from the starting line, a volcano blowing its lava into the atmosphere, or a rocket blasting off, as in **Figure 7.23**.

1. Representative values

Figure 7.23 This powerful rocket on the Space Shuttle *Endeavor* did work and consumed energy at a very high rate. (credit: NASA)

These images of power have in common the rapid performance of work, consistent with the scientific definition of **power** (P) as the rate at which work is done.

Power

Power is the rate at which work is done.

$$P = \frac{W}{t} \tag{7.69}$$

The SI unit for power is the **watt** (W), where 1 watt equals 1 joule/second $(1\ \text{W} = 1\ \text{J/s})$.

Because work is energy transfer, power is also the rate at which energy is expended. A 60-W light bulb, for example, expends 60 J of energy per second. Great power means a large amount of work or energy developed in a short time. For example, when a powerful car accelerates rapidly, it does a large amount of work and consumes a large amount of fuel in a short time.

Calculating Power from Energy

Example 7.11 Calculating the Power to Climb Stairs

What is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s? (See **Figure 7.24**.)

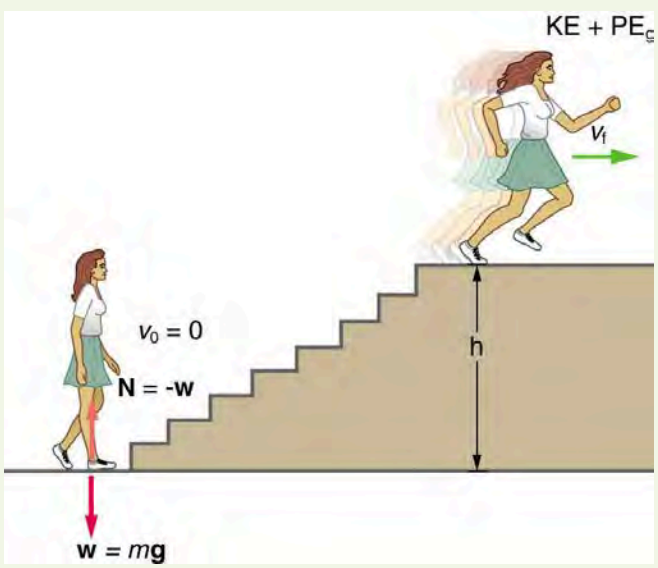

Figure 7.24 When this woman runs upstairs starting from rest, she converts the chemical energy originally from food into kinetic energy and gravitational potential energy. Her power output depends on how fast she does this.

Strategy and Concept

The work going into mechanical energy is $W = \text{KE} + \text{PE}$. At the bottom of the stairs, we take both KE and PE_g as initially zero; thus, $W = \text{KE}_\text{f} + \text{PE}_\text{g} = \frac{1}{2}mv_\text{f}^2 + mgh$, where h is the vertical height of the stairs. Because all terms are given, we can calculate W and then divide it by time to get power.

Solution

Substituting the expression for W into the definition of power given in the previous equation, $P = W/t$ yields

$$P = \frac{W}{t} = \frac{\frac{1}{2}mv_\text{f}^2 + mgh}{t}. \tag{7.70}$$

Entering known values yields

$$\begin{aligned} P &= \frac{0.5(60.0\text{ kg})(2.00\text{ m/s})^2 + (60.0\text{ kg})\left(9.80\text{ m/s}^2\right)(3.00\text{ m})}{3.50\text{ s}} \\ &= \frac{120\text{ J} + 1764\text{ J}}{3.50\text{ s}} \\ &= 538\text{ W}. \end{aligned} \tag{7.71}$$

Discussion

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.

It is impressive that this woman's useful power output is slightly less than 1 **horsepower** $(1\text{ hp} = 746\text{ W})$! People can generate more than a horsepower with their leg muscles for short periods of time by rapidly converting available blood sugar and oxygen into work output. (A horse can put out 1 hp for hours on end.) Once oxygen is depleted, power output decreases and the person begins to breathe rapidly to obtain oxygen to metabolize more food—this is known as the *aerobic* stage of exercise. If the woman climbed the stairs slowly, then her power output would be much less, although the amount of work done would be the same.

Making Connections: Take-Home Investigation—Measure Your Power Rating

Determine your own power rating by measuring the time it takes you to climb a flight of stairs. We will ignore the gain in kinetic energy, as the above example showed that it was a small portion of the energy gain. Don't expect that your output will be more than about 0.5 hp.

Examples of Power

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. (See **Table 7.3** for some examples.) Sunlight reaching Earth's surface carries a maximum power of about 1.3 kilowatts per square meter (kW/m^2). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40% of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1000 megawatts; 1 megawatt (MW) is 10^6 W of electric power. But the power plant consumes chemical energy at a rate of about 2500 MW, creating heat transfer to the surroundings at a rate of 1500 MW. (See **Figure 7.25**.)

Figure 7.25 Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger amount of power goes into heat transfer to the surroundings. The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like. (credit: Kleinolive, Wikimedia Commons)

Table 7.3 Power Output or Consumption

Object or Phenomenon	Power in Watts
Supernova (at peak)	5×10^{37}
Milky Way galaxy	10^{37}
Crab Nebula pulsar	10^{28}
The Sun	4×10^{26}
Volcanic eruption (maximum)	4×10^{15}
Lightning bolt	2×10^{12}
Nuclear power plant (total electric and heat transfer)	3×10^{9}
Aircraft carrier (total useful and heat transfer)	10^{8}
Dragster (total useful and heat transfer)	2×10^{6}
Car (total useful and heat transfer)	8×10^{4}
Football player (total useful and heat transfer)	5×10^{3}
Clothes dryer	4×10^{3}
Person at rest (all heat transfer)	100
Typical incandescent light bulb (total useful and heat transfer)	60
Heart, person at rest (total useful and heat transfer)	8
Electric clock	3
Pocket calculator	10^{-3}

Power and Energy Consumption

We usually have to pay for the energy we use. It is interesting and easy to estimate the cost of energy for an electrical appliance if its power consumption rate and time used are known. The higher the power consumption rate and the longer the appliance is used, the greater the cost of that appliance. The power consumption rate is $P = W/t = E/t$, where E is the energy supplied by the electricity company. So the energy consumed over a time t is

$$E = Pt. \tag{7.72}$$

Electricity bills state the energy used in units of **kilowatt-hours** $(\mathrm{kW \cdot h})$, which is the product of power in kilowatts and time in hours. This unit is convenient because electrical power consumption at the kilowatt level for hours at a time is typical.

Example 7.12 Calculating Energy Costs

What is the cost of running a 0.200-kW computer 6.00 h per day for 30.0 d if the cost of electricity is \$0.120 per $\mathrm{kW \cdot h}$?

Strategy

Cost is based on energy consumed; thus, we must find E from $E = Pt$ and then calculate the cost. Because electrical energy is expressed in $\mathrm{kW \cdot h}$, at the start of a problem such as this it is convenient to convert the units into kW and hours.

Solution

The energy consumed in $\mathrm{kW \cdot h}$ is

$$\begin{aligned} E &= Pt = (0.200\ \mathrm{kW})(6.00\ \mathrm{h/d})(30.0\ \mathrm{d}) \\ &= 36.0\ \mathrm{kW \cdot h}, \end{aligned} \tag{7.73}$$

and the cost is simply given by

$$\text{cost} = (36.0\ \mathrm{kW \cdot h})(\$0.120\ \text{per}\ \mathrm{kW \cdot h}) = \$4.32\ \text{per month}. \tag{7.74}$$

Discussion

The cost of using the computer in this example is neither exorbitant nor negligible. It is clear that the cost is a combination of power and time. When both are high, such as for an air conditioner in the summer, the cost is high.

The motivation to save energy has become more compelling with its ever-increasing price. Armed with the knowledge that energy consumed is the product of power and time, you can estimate costs for yourself and make the necessary value judgments about where to save energy. Either power or time must be reduced. It is most cost-effective to limit the use of high-power devices that normally operate for long periods of time, such as water heaters and air conditioners. This would not include relatively high power devices like toasters, because they are on only a few minutes per day. It would also not include electric clocks, in spite of their 24-hour-per-day usage, because they are very low power devices. It is sometimes possible to use devices that have greater efficiencies—that is, devices that consume less power to accomplish the same task. One example is the compact fluorescent light bulb, which produces over four times more light per watt of power consumed than its incandescent cousin.

Modern civilization depends on energy, but current levels of energy consumption and production are not sustainable. The likelihood of a link between global warming and fossil fuel use (with its concomitant production of carbon dioxide), has made reduction in energy use as well as a shift to non-fossil fuels of the utmost importance. Even though energy in an isolated system is a conserved quantity, the final result of most energy transformations is waste heat transfer to the environment, which is no longer useful for doing work. As we will discuss in more detail in **Thermodynamics**, the potential for energy to produce useful work has been "degraded" in the energy transformation.

7.8 Work, Energy, and Power in Humans

Learning Objectives

By the end of this section, you will be able to:

- Explain the human body's consumption of energy when at rest versus when engaged in activities that do useful work.
- Calculate the conversion of chemical energy in food into useful work.

Energy Conversion in Humans

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, and/or stored as chemical energy in fatty tissue. (See **Figure 7.26**.) The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.

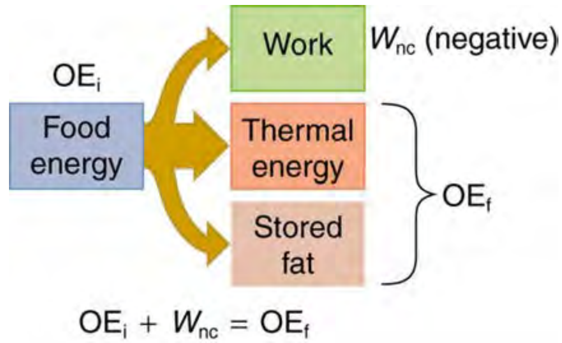

Figure 7.26 Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.

Power Consumed at Rest

The *rate* at which the body uses food energy to sustain life and to do different activities is called the **metabolic rate**. The total energy conversion rate of a person *at rest* is called the **basal metabolic rate** (BMR) and is divided among various systems in the body, as shown in **Table 7.4**. The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.

Table 7.4 Basal Metabolic Rates (BMR)

Organ	Power consumed at rest (W)	Oxygen consumption (mL/min)	Percent of BMR
Liver & spleen	23	67	27
Brain	16	47	19
Skeletal muscle	15	45	18
Kidney	9	26	10
Heart	6	17	7
Other	16	48	19
Totals	85 W	250 mL/min	100%

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. (See **Figure 7.27**.) Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food. **Table 7.5** shows energy and oxygen consumption rates (power expended) for a variety of activities.

Power of Doing Useful Work

Work done by a person is sometimes called **useful work**, which is *work done on the outside world*, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the mechanical energy ($\mathrm{KE} + \mathrm{PE}$) of the system worked upon, and this is often the goal. A baseball player throwing a ball, for example, increases both the ball's kinetic and potential energy.

If a person needs more energy than they consume, such as when doing vigorous work, the body must draw upon the chemical energy stored in fat. So exercise can be helpful in losing fat. However, the amount of exercise needed to produce a loss in fat, or to burn off extra calories consumed that day, can be large, as **Example 7.13** illustrates.

Example 7.13 Calculating Weight Loss from Exercising

If a person who normally requires an average of 12,000 kJ (3000 kcal) of food energy per day consumes 13,000 kJ per day, he will steadily gain weight. How much bicycling per day is required to work off this extra 1000 kJ?

Solution

Table 7.5 states that 400 W are used when cycling at a moderate speed. The time required to work off 1000 kJ at this rate is then

$$\text{Time} = \frac{\text{energy}}{\left(\frac{\text{energy}}{\text{time}}\right)} = \frac{1000\ \text{kJ}}{400\ \text{W}} = 2500\ \text{s} = 42\ \text{min}. \tag{7.75}$$

Discussion

If this person uses more energy than he or she consumes, the person's body will obtain the needed energy by metabolizing body fat. If the person uses 13,000 kJ but consumes only 12,000 kJ, then the amount of fat loss will be

$$\text{Fat loss} = (1000\text{ kJ})\left(\frac{1.0\text{ g fat}}{39\text{ kJ}}\right) = 26\text{ g}, \tag{7.76}$$

assuming the energy content of fat to be 39 kJ/g.

Figure 7.27 A pulse oxymeter is an apparatus that measures the amount of oxygen in blood. Oxymeters can be used to determine a person's metabolic rate, which is the rate at which food energy is converted to another form. Such measurements can indicate the level of athletic conditioning as well as certain medical problems. (credit: UusiAjaja, Wikimedia Commons)

Table 7.5 Energy and Oxygen Consumption Rates[2] (Power)

Activity	Energy consumption in watts	Oxygen consumption in liters O_2/min
Sleeping	83	0.24
Sitting at rest	120	0.34
Standing relaxed	125	0.36
Sitting in class	210	0.60
Walking (5 km/h)	280	0.80
Cycling (13–18 km/h)	400	1.14
Shivering	425	1.21
Playing tennis	440	1.26
Swimming breaststroke	475	1.36
Ice skating (14.5 km/h)	545	1.56
Climbing stairs (116/min)	685	1.96
Cycling (21 km/h)	700	2.00
Running cross-country	740	2.12
Playing basketball	800	2.28
Cycling, professional racer	1855	5.30
Sprinting	2415	6.90

All bodily functions, from thinking to lifting weights, require energy. (See **Figure 7.28**.) The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract. Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all it that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.

2. for an average 76-kg male

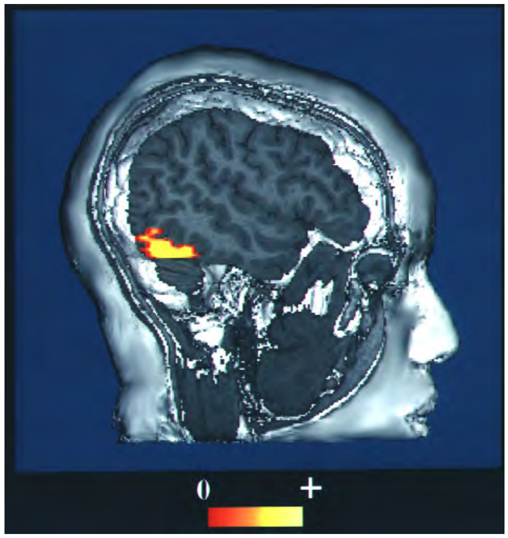

Figure 7.28 This fMRI scan shows an increased level of energy consumption in the vision center of the brain. Here, the patient was being asked to recognize faces. (credit: NIH via Wikimedia Commons)

7.9 World Energy Use

Learning Objectives

By the end of this section, you will be able to:

- Describe the distinction between renewable and nonrenewable energy sources.
- Explain why the inevitable conversion of energy to less useful forms makes it necessary to conserve energy resources.

Energy is an important ingredient in all phases of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. But current levels of energy consumption and production are not sustainable. About 40% of the world's energy comes from oil, and much of that goes to transportation uses. Oil prices are dependent as much upon new (or foreseen) discoveries as they are upon political events and situations around the world. The U.S., with 4.5% of the world's population, consumes 24% of the world's oil production per year; 66% of that oil is imported!

Renewable and Nonrenewable Energy Sources

The principal energy resources used in the world are shown in **Figure 7.29**. The fuel mix has changed over the years but now is dominated by oil, although natural gas and solar contributions are increasing. **Renewable forms of energy** are those sources that cannot be used up, such as water, wind, solar, and biomass. About 85% of our energy comes from nonrenewable **fossil fuels**—oil, natural gas, coal. The likelihood of a link between global warming and fossil fuel use, with its production of carbon dioxide through combustion, has made, in the eyes of many scientists, a shift to non-fossil fuels of utmost importance—but it will not be easy.

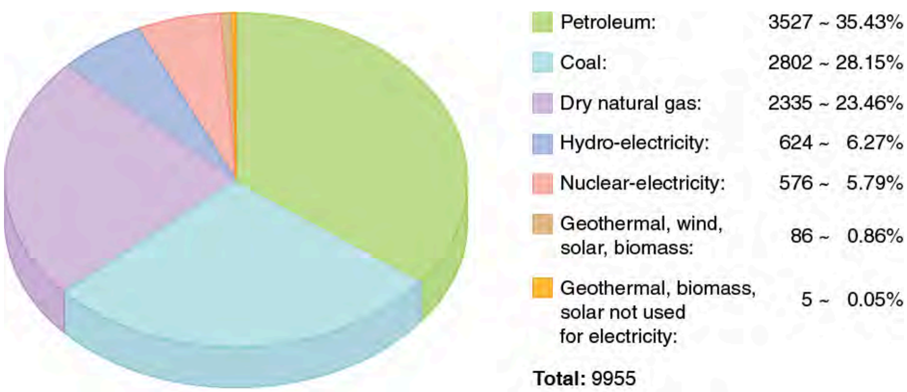

Figure 7.29 World energy consumption by source, in billions of kilowatt-hours: 2006. (credit: KVDP)

The World's Growing Energy Needs

World energy consumption continues to rise, especially in the developing countries. (See **Figure 7.30**.) Global demand for energy has tripled in the past 50 years and might triple again in the next 30 years. While much of this growth will come from the rapidly booming economies of China and India, many of the developed countries, especially those in Europe, are hoping to meet their energy needs by expanding the use of renewable sources. Although presently only a small percentage, renewable energy is growing very fast, especially wind energy. For example, Germany plans to meet 20% of its electricity and 10% of its overall energy needs with renewable resources by the year 2020. (See **Figure 7.31**.) Energy is a key constraint in the rapid economic growth of China and India. In 2003, China surpassed Japan as the world's second largest consumer of oil. However, over 1/3 of this is imported. Unlike most Western countries, coal dominates the commercial energy resources of China, accounting for 2/3 of its energy consumption. In 2009 China surpassed the United States as the largest generator of CO_2 . In India, the main energy resources are biomass (wood and dung) and coal. Half of India's oil is imported. About 70% of India's electricity is generated by highly polluting coal. Yet there are sizeable strides being made in renewable energy. India has a rapidly growing wind energy base, and it has the largest solar cooking program in the world.

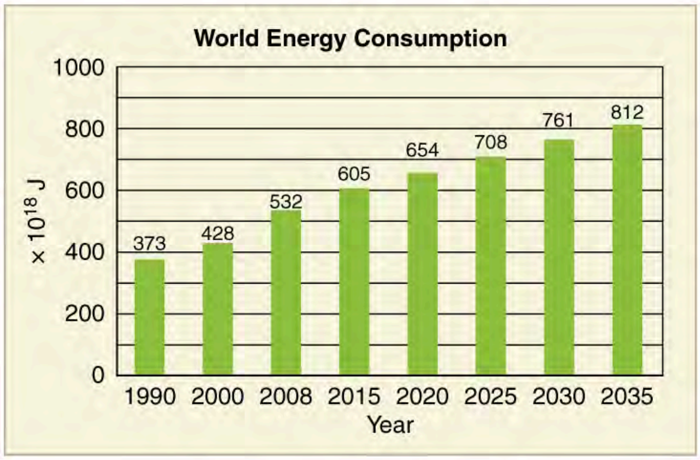

Figure 7.30 Past and projected world energy use (source: Based on data from U.S. Energy Information Administration, 2011)

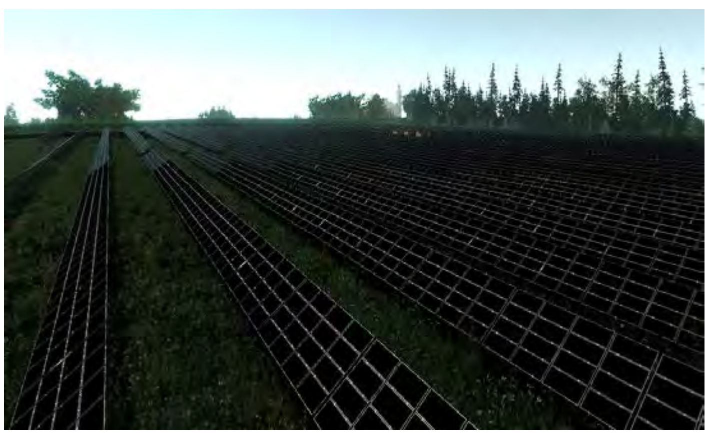

Figure 7.31 Solar cell arrays at a power plant in Steindorf, Germany (credit: Michael Betke, Flickr)

Table 7.6 displays the 2006 commercial energy mix by country for some of the prime energy users in the world. While non-renewable sources dominate, some countries get a sizeable percentage of their electricity from renewable resources. For example, about 67% of New Zealand's electricity demand is met by hydroelectric. Only 10% of the U.S. electricity is generated by renewable resources, primarily hydroelectric. It is difficult to determine total contributions of renewable energy in some countries with a large rural population, so these percentages in this table are left blank.

Table 7.6 Energy Consumption—Selected Countries (2006)

Country	Consumption, in EJ (10^{18} J)	Oil	Natural Gas	Coal	Nuclear	Hydro	Other Renewables	Electricity Use per capita (kWh/yr)	Energy Use per capita (GJ/yr)
Australia	5.4	34%	17%	44%	0%	3%	1%	10000	260
Brazil	9.6	48%	7%	5%	1%	35%	2%	2000	50
China	63	22%	3%	69%	1%	6%		1500	35
Egypt	2.4	50%	41%	1%	0%	6%		990	32
Germany	16	37%	24%	24%	11%	1%	3%	6400	173
India	15	34%	7%	52%	1%	5%		470	13
Indonesia	4.9	51%	26%	16%	0%	2%	3%	420	22
Japan	24	48%	14%	21%	12%	4%	1%	7100	176
New Zealand	0.44	32%	26%	6%	0%	11%	19%	8500	102
Russia	31	19%	53%	16%	5%	6%		5700	202
U.S.	105	40%	23%	22%	8%	3%	1%	12500	340
World	**432**	**39%**	**23%**	**24%**	**6%**	**6%**	**2%**	**2600**	**71**

Energy and Economic Well-being

The last two columns in this table examine the energy and electricity use per capita. Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (gross domestic product) per capita, are matched by higher levels of energy consumption per capita. This is borne out in **Figure 7.32**. Increased efficiency of energy use will change this dependency. A global problem is balancing energy resource development against the harmful effects upon the environment in its extraction and use.

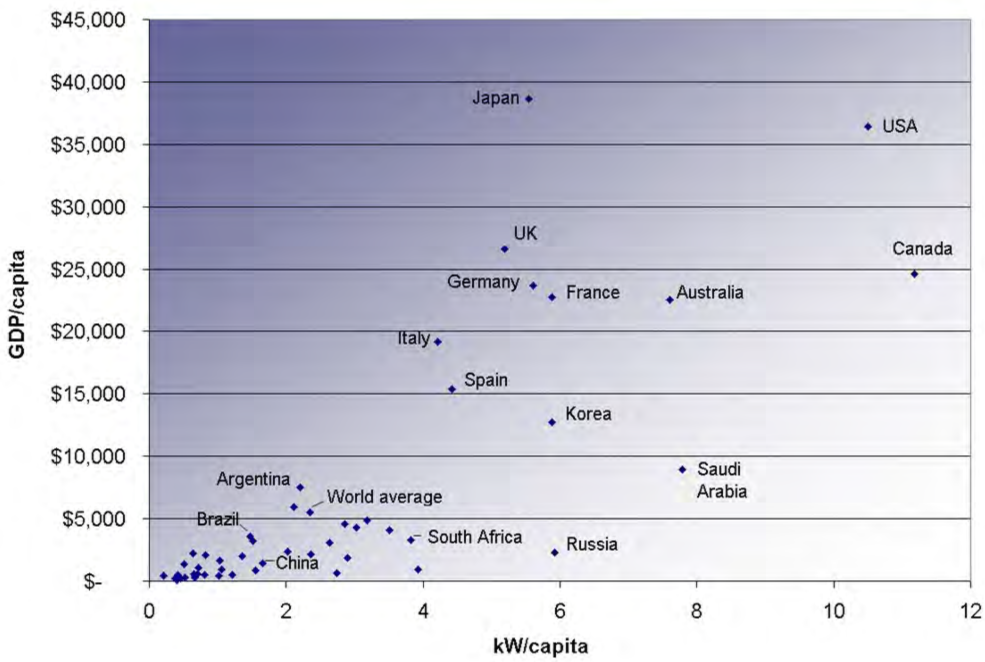

Figure 7.32 Power consumption per capita versus GDP per capita for various countries. Note the increase in energy usage with increasing GDP. (2007, credit: Frank van Mierlo, Wikimedia Commons)

Conserving Energy

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As has been mentioned elsewhere, the “law of the conservation of energy” is a very useful principle in analyzing physical processes. It is a statement that cannot be proven from basic principles, but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system will always remain constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through (1) reduced activities (e.g., turning down thermostats, driving fewer kilometers) and/or (2) increasing conversion efficiencies in the performance of a particular task—such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed or created or generated, one might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat transfer to the environment and conversion to energy forms no longer useful for doing work. To state it in another way, the potential for energy to produce useful work has been "degraded" in the energy transformation. (This will be discussed in more detail in **Thermodynamics**.)

Glossary

basal metabolic rate: the total energy conversion rate of a person at rest

chemical energy: the energy in a substance stored in the bonds between atoms and molecules that can be released in a chemical reaction

conservation of mechanical energy: the rule that the sum of the kinetic energies and potential energies remains constant if only conservative forces act on and within a system

conservative force: a force that does the same work for any given initial and final configuration, regardless of the path followed

efficiency: a measure of the effectiveness of the input of energy to do work; useful energy or work divided by the total input of energy

electrical energy: the energy carried by a flow of charge

energy: the ability to do work

fossil fuels: oil, natural gas, and coal

friction: the force between surfaces that opposes one sliding on the other; friction changes mechanical energy into thermal energy

gravitational potential energy: the energy an object has due to its position in a gravitational field

horsepower: an older non-SI unit of power, with $1 \text{ hp} = 746 \text{ W}$

joule: SI unit of work and energy, equal to one newton-meter

kilowatt-hour: $(\text{kW} \cdot \text{h})$ unit used primarily for electrical energy provided by electric utility companies

kinetic energy: the energy an object has by reason of its motion, equal to $\frac{1}{2}mv^2$ for the translational (i.e., non-rotational) motion of an object of mass m moving at speed v

law of conservation of energy: the general law that total energy is constant in any process; energy may change in form or be transferred from one system to another, but the total remains the same

mechanical energy: the sum of kinetic energy and potential energy

metabolic rate: the rate at which the body uses food energy to sustain life and to do different activities

net work: work done by the net force, or vector sum of all the forces, acting on an object

nonconservative force: a force whose work depends on the path followed between the given initial and final configurations

nuclear energy: energy released by changes within atomic nuclei, such as the fusion of two light nuclei or the fission of a heavy nucleus

potential energy: energy due to position, shape, or configuration

potential energy of a spring: the stored energy of a spring as a function of its displacement; when Hooke's law applies, it is given by the expression $\frac{1}{2}kx^2$ where x is the distance the spring is compressed or extended and k is the spring constant

power: the rate at which work is done

radiant energy: the energy carried by electromagnetic waves

renewable forms of energy: those sources that cannot be used up, such as water, wind, solar, and biomass

thermal energy: the energy within an object due to the random motion of its atoms and molecules that accounts for the object's temperature

useful work: work done on an external system

watt: (W) SI unit of power, with $1\ \text{W} = 1\ \text{J/s}$

work: the transfer of energy by a force that causes an object to be displaced; the product of the component of the force in the direction of the displacement and the magnitude of the displacement

work-energy theorem: the result, based on Newton's laws, that the net work done on an object is equal to its change in kinetic energy

Section Summary

7.1 Work: The Scientific Definition

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work W that a force $\mathbf{F}$ does on an object is the product of the magnitude F of the force, times the magnitude d of the displacement, times the cosine of the angle θ between them. In symbols,

$$W = Fd\cos\theta.$$

- The SI unit for work and energy is the joule (J), where $1\ \text{J} = 1\ \text{N}\cdot\text{m} = 1\ \text{kg}\cdot\text{m}^2/\text{s}^2$.
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.

7.2 Kinetic Energy and the Work-Energy Theorem

- The net work W_{net} is the work done by the net force acting on an object.
- Work done on an object transfers energy to the object.
- The translational kinetic energy of an object of mass m moving at speed v is $\text{KE} = \frac{1}{2}mv^2$.
- The work-energy theorem states that the net work W_{net} on a system changes its kinetic energy, $W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2$.

7.3 Gravitational Potential Energy

- Work done against gravity in lifting an object becomes potential energy of the object-Earth system.
- The change in gravitational potential energy, ΔPE_g, is $\Delta\text{PE}_\text{g} = mgh$, with h being the increase in height and g the acceleration due to gravity.
- The gravitational potential energy of an object near Earth's surface is due to its position in the mass-Earth system. Only differences in gravitational potential energy, ΔPE_g, have physical significance.
- As an object descends without friction, its gravitational potential energy changes into kinetic energy corresponding to increasing speed, so that $\Delta\text{KE} = -\Delta\text{PE}_\text{g}$.

7.4 Conservative Forces and Potential Energy

- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy (PE) for any conservative force, just as we defined PE_g for the gravitational force.
- The potential energy of a spring is $\text{PE}_\text{s} = \frac{1}{2}kx^2$, where k is the spring's force constant and x is the displacement from its undeformed position.
- Mechanical energy is defined to be $\text{KE} + \text{PE}$ for a conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,

$$\left.\begin{array}{c}\mathrm{KE}+\mathrm{PE}=\text{constant}\\ \text{or}\\ \mathrm{KE_i}+\mathrm{PE_i}=\mathrm{KE_f}+\mathrm{PE_f}\end{array}\right\}$$

where i and f denote initial and final values. This is known as the conservation of mechanical energy.

7.5 Nonconservative Forces

- A nonconservative force is one for which work depends on the path.
- Friction is an example of a nonconservative force that changes mechanical energy into thermal energy.
- Work W_{nc} done by a nonconservative force changes the mechanical energy of a system. In equation form, $W_{\mathrm{nc}}=\Delta\mathrm{KE}+\Delta\mathrm{PE}$ or, equivalently, $\mathrm{KE_i}+\mathrm{PE_i}+W_{\mathrm{nc}}=\mathrm{KE_f}+\mathrm{PE_f}$.
- When both conservative and nonconservative forces act, energy conservation can be applied and used to calculate motion in terms of the known potential energies of the conservative forces and the work done by nonconservative forces, instead of finding the net work from the net force, or having to directly apply Newton's laws.

7.6 Conservation of Energy

- The law of conservation of energy states that the total energy is constant in any process. Energy may change in form or be transferred from one system to another, but the total remains the same.
- When all forms of energy are considered, conservation of energy is written in equation form as $\mathrm{KE_i}+\mathrm{PE_i}+W_{\mathrm{nc}}+\mathrm{OE_i}=\mathrm{KE_f}+\mathrm{PE_f}+\mathrm{OE_f}$, where OE is all **other forms of energy** besides mechanical energy.
- Commonly encountered forms of energy include electric energy, chemical energy, radiant energy, nuclear energy, and thermal energy.
- Energy is often utilized to do work, but it is not possible to convert all the energy of a system to work.
- The efficiency Eff of a machine or human is defined to be $\mathit{Eff}=\frac{W_{\mathrm{out}}}{E_{\mathrm{in}}}$, where W_{out} is useful work output and E_{in} is the energy consumed.

7.7 Power

- Power is the rate at which work is done, or in equation form, for the average power P for work W done over a time t, $P=W/t$.
- The SI unit for power is the watt (W), where $1\ \mathrm{W}=1\ \mathrm{J/s}$.
- The power of many devices such as electric motors is also often expressed in horsepower (hp), where $1\ \mathrm{hp}=746\ \mathrm{W}$.

7.8 Work, Energy, and Power in Humans

- The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.
- The *rate* at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR)
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.
- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

7.9 World Energy Use

- The relative use of different fuels to provide energy has changed over the years, but fuel use is currently dominated by oil, although natural gas and solar contributions are increasing.
- Although non-renewable sources dominate, some countries meet a sizeable percentage of their electricity needs from renewable resources.
- The United States obtains only about 10% of its energy from renewable sources, mostly hydroelectric power.
- Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (Gross Domestic Product) per capita, are matched by higher levels of energy consumption per capita.
- Even though, in accordance with the law of conservation of energy, energy can never be created or destroyed, energy that can be used to do work is always partly converted to less useful forms, such as waste heat to the environment, in all of our uses of energy for practical purposes.

Conceptual Questions

7.1 Work: The Scientific Definition

1. Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

2. Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

3. Describe a situation in which a force is exerted for a long time but does no work. Explain.

7.2 Kinetic Energy and the Work-Energy Theorem

4. The person in **Figure 7.33** does work on the lawn mower. Under what conditions would the mower gain energy? Under what conditions would it lose energy?

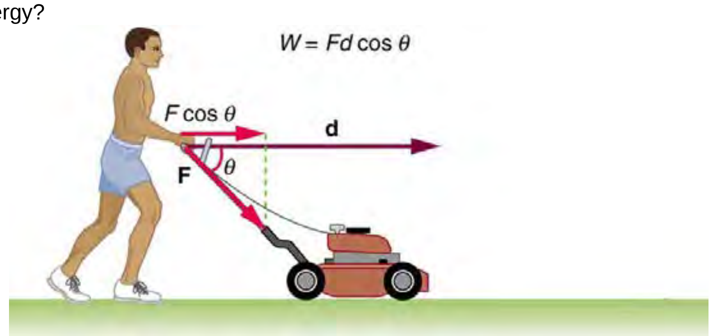

Figure 7.33

5. Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

6. When solving for speed in **Example 7.4**, we kept only the positive root. Why?

7.3 Gravitational Potential Energy

7. In **Example 7.7**, we calculated the final speed of a roller coaster that descended 20 m in height and had an initial speed of 5 m/s downhill. Suppose the roller coaster had had an initial speed of 5 m/s uphill instead, and it coasted uphill, stopped, and then rolled back down to a final point 20 m below the start. We would find in that case that its final speed is the same as its initial. Explain in terms of conservation of energy.

8. Does the work you do on a book when you lift it onto a shelf depend on the path taken? On the time taken? On the height of the shelf? On the mass of the book?

7.4 Conservative Forces and Potential Energy

9. What is a conservative force?

10. The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer dives from it, starting just before the swimmer steps on the board until just after his feet leave it.

11. Define mechanical energy. What is the relationship of mechanical energy to nonconservative forces? What happens to mechanical energy if only conservative forces act?

12. What is the relationship of potential energy to conservative force?

7.6 Conservation of Energy

13. Consider the following scenario. A car for which friction is *not* negligible accelerates from rest down a hill, running out of gasoline after a short distance. The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events. (See **Figure 7.34**.)

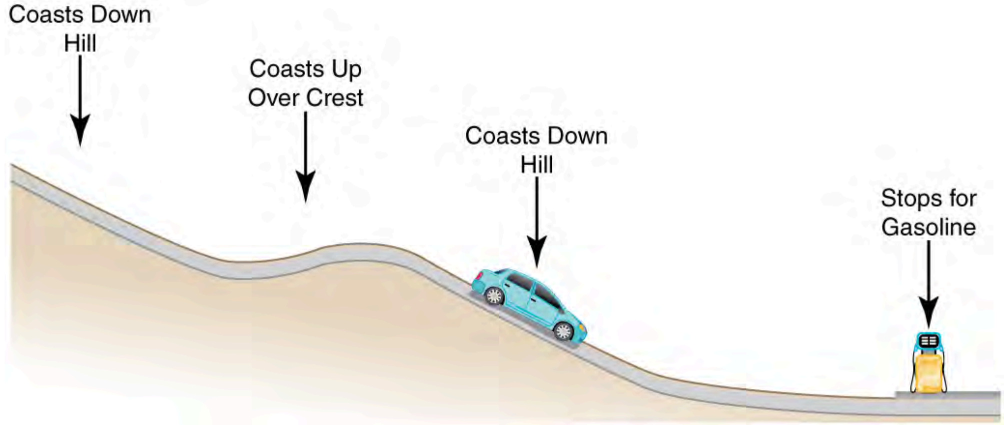

Figure 7.34 A car experiencing non-negligible friction coasts down a hill, over a small crest, then downhill again, and comes to a stop at a gas station.

14. Describe the energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

15. Do devices with efficiencies of less than one violate the law of conservation of energy? Explain.

16. List four different forms or types of energy. Give one example of a conversion from each of these forms to another form.

17. List the energy conversions that occur when riding a bicycle.

7.7 Power

18. Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

19. Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

20. A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

7.8 Work, Energy, and Power in Humans

21. Explain why it is easier to climb a mountain on a zigzag path rather than one straight up the side. Is your increase in gravitational potential energy the same in both cases? Is your energy consumption the same in both?

22. Do you do work on the outside world when you rub your hands together to warm them? What is the efficiency of this activity?

23. Shivering is an involuntary response to lowered body temperature. What is the efficiency of the body when shivering, and is this a desirable value?

24. Discuss the relative effectiveness of dieting and exercise in losing weight, noting that most athletic activities consume food energy at a rate of 400 to 500 W, while a single cup of yogurt can contain 1360 kJ (325 kcal). Specifically, is it likely that exercise alone will be sufficient to lose weight? You may wish to consider that regular exercise may increase the metabolic rate, whereas protracted dieting may reduce it.

7.9 World Energy Use

25. What is the difference between energy conservation and the law of conservation of energy? Give some examples of each.

26. If the efficiency of a coal-fired electrical generating plant is 35%, then what do we mean when we say that energy is a conserved quantity?

Problems & Exercises

7.1 Work: The Scientific Definition

1. How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

2. A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

3. (a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

4. Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See **Table 7.1** for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

5. Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0° with the horizontal. (See **Figure 7.35**.) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate *and* on his body to get up the ramp.

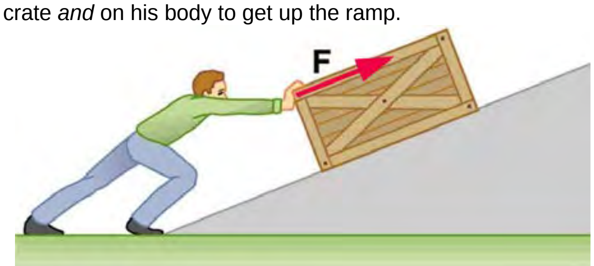

Figure 7.35 A man pushes a crate up a ramp.

6. How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in **Figure 7.36**? Assume no friction acts on the wagon.

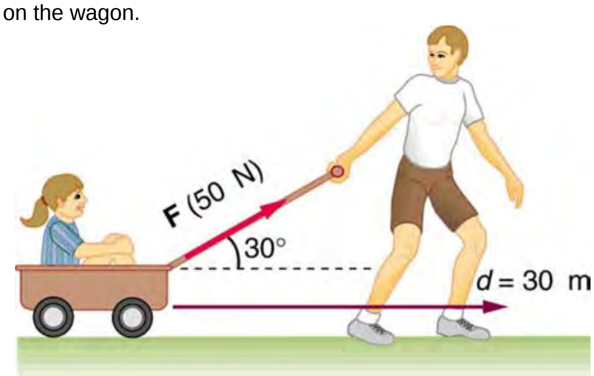

Figure 7.36 The boy does work on the system of the wagon and the child when he pulls them as shown.

7. A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction 25.0° below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

8. Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0° slope at constant speed, as shown in **Figure 7.37**. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?

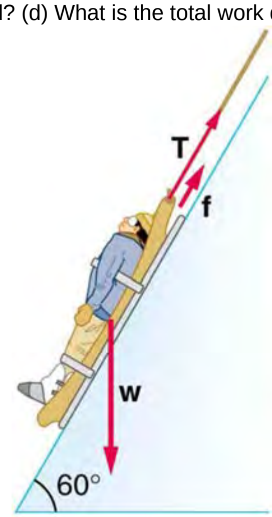

Figure 7.37 A rescue sled and victim are lowered down a steep slope.

7.2 Kinetic Energy and the Work-Energy Theorem

9. Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

10. (a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

11. Confirm the value given for the kinetic energy of an aircraft carrier in **Table 7.1**. You will need to look up the definition of a nautical mile (1 knot = 1 nautical mile/h).

12. (a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

13. A car's bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.

14. Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent's face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the gory old days when no gloves were used and the knuckles and face would compress only 2.00 cm. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

15. Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

7.3 Gravitational Potential Energy

16. A hydroelectric power facility (see **Figure 7.38**) converts the gravitational potential energy of water behind a dam to electric energy. (a) What is the gravitational potential energy relative to the generators of a lake of volume $50.0\ \text{km}^3$ ($\text{mass} = 5.00\times10^{13}\ \text{kg})$, given that the lake has an average height of 40.0 m above the generators? (b) Compare this with the energy stored in a 9-megaton fusion bomb.

Figure 7.38 Hydroelectric facility (credit: Denis Belevich, Wikimedia Commons)

17. (a) How much gravitational potential energy (relative to the ground on which it is built) is stored in the Great Pyramid of Cheops, given that its mass is about $7\times10^9\ \text{kg}$ and its center of mass is 36.5 m above the surrounding ground? (b) How does this energy compare with the daily food intake of a person?

18. Suppose a 350-g kookaburra (a large kingfisher bird) picks up a 75-g snake and raises it 2.5 m from the ground to a branch. (a) How much work did the bird do on the snake? (b) How much work did it do to raise its own center of mass to the branch?

19. In **Example 7.7**, we found that the speed of a roller coaster that had descended 20.0 m was only slightly greater when it had an initial speed of 5.00 m/s than when it started from rest. This implies that $\Delta\text{PE} >> \text{KE}_\text{i}$. Confirm this statement by taking the ratio of ΔPE to KE_i . (Note that mass cancels.)

20. A 100-g toy car is propelled by a compressed spring that starts it moving. The car follows the curved track in **Figure 7.39**. Show that the final speed of the toy car is 0.687 m/s if its initial speed is 2.00 m/s and it coasts up the frictionless slope, gaining 0.180 m in altitude.

Figure 7.39 A toy car moves up a sloped track. (credit: Leszek Leszczynski, Flickr)

21. In a downhill ski race, surprisingly, little advantage is gained by getting a running start. (This is because the initial kinetic energy is small compared with the gain in gravitational potential energy on even small hills.) To demonstrate this, find the final speed and the time taken for a skier who skies 70.0 m along a 30° slope neglecting friction: (a) Starting from rest. (b) Starting with an initial speed of 2.50 m/s. (c) Does the answer surprise you? Discuss why it is still advantageous to get a running start in very competitive events.

7.4 Conservative Forces and Potential Energy

22. A $5.00\times10^5\text{-kg}$ subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the force constant k of the spring?

23. A pogo stick has a spring with a force constant of $2.50\times10^4\ \text{N/m}$, which can be compressed 12.0 cm. To what maximum height can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40.0 kg? Explicitly show how you follow the steps in the **Problem-Solving Strategies for Energy**.

7.5 Nonconservative Forces

24. A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m-high rise as shown in **Figure 7.40**. Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.0800. (Hint: Find the distance traveled up the incline assuming a straight-line path as shown in the figure.)

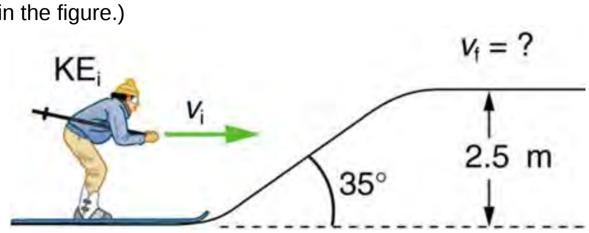

Figure 7.40 The skier's initial kinetic energy is partially used in coasting to the top of a rise.

25. (a) How high a hill can a car coast up (engine disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope 2.5° above the horizontal?

7.6 Conservation of Energy

26. Using values from **Table 7.1**, how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous x rays. Later model tube TVs had shielding that absorbed x rays before they escaped and exposed viewers.)

27. Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown.

28. If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from **Table 7.1**)? This is not as far-fetched as it may sound—there are thousands of nuclear bombs, and their energy can be trapped in underground explosions and converted to electricity, as natural geothermal energy is.

29. (a) Use of hydrogen fusion to supply energy is a dream that may be realized in the next century. Fusion would be a relatively clean and almost limitless supply of energy, as can be seen from **Table 7.1**. To illustrate this, calculate how many years the present energy needs of the world could be supplied by one millionth of the oceans' hydrogen fusion energy. (b) How does this time compare with historically significant events, such as the duration of stable economic systems?

7.7 Power

30. The Crab Nebula (see **Figure 7.41**) pulsar is the remnant of a supernova that occurred in A.D. 1054. Using data from **Table 7.3**, calculate the approximate factor by which the power output of this astronomical object has declined since its explosion.

Figure 7.41 Crab Nebula (credit: ESO, via Wikimedia Commons)

31. Suppose a star 1000 times brighter than our Sun (that is, emitting 1000 times the power) suddenly goes supernova. Using data from **Table 7.3**: (a) By what factor does its power output increase? (b) How many times brighter than our entire Milky Way galaxy is the supernova? (c) Based on your answers, discuss whether it should be possible to observe supernovas in distant galaxies. Note that there are on the order of 10^{11} observable galaxies, the average brightness of which is somewhat less than our own galaxy.

32. A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

33. What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is $0.0900 per $\text{kW} \cdot \text{h}$?

34. A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is $0.110 per $\text{kW} \cdot \text{h}$?

35. (a) What is the average power consumption in watts of an appliance that uses $5.00\ \text{kW} \cdot \text{h}$ of energy per day? (b) How many joules of energy does this appliance consume in a year?

36. (a) What is the average useful power output of a person who does $6.00\times10^6\ \text{J}$ of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

37. A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?

38. (a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp = 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m-high hill in the process?

39. (a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is $0.0900 per $\mathrm{kW \cdot h}$?

40. (a) What is the available energy content, in joules, of a battery that operates a 2.00-W electric clock for 18 months? (b) How long can a battery that can supply $8.00 \times 10^4\ \mathrm{J}$ run a pocket calculator that consumes energy at the rate of $1.00 \times 10^{-3}\ \mathrm{W}$?

41. (a) How long would it take a 1.50×10^5-kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (Hint: You must find the distance the plane travels in 1200 s assuming constant acceleration.)

42. Calculate the power output needed for a 950-kg car to climb a 2.00° slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N. Explicitly show how you follow the steps in the **Problem-Solving Strategies for Energy**.

43. (a) Calculate the power per square meter reaching Earth's upper atmosphere from the Sun. (Take the power output of the Sun to be $4.00 \times 10^{26}\ \mathrm{W}$.) (b) Part of this is absorbed and reflected by the atmosphere, so that a maximum of $1.30\ \mathrm{kW/m^2}$ reaches Earth's surface. Calculate the area in $\mathrm{km^2}$ of solar energy collectors needed to replace an electric power plant that generates 750 MW if the collectors convert an average of 2.00% of the maximum power into electricity. (This small conversion efficiency is due to the devices themselves, and the fact that the sun is directly overhead only briefly.) With the same assumptions, what area would be needed to meet the United States' energy needs $(1.05 \times 10^{20}\ \mathrm{J})$? Australia's energy needs $(5.4 \times 10^{18}\ \mathrm{J})$? China's energy needs $(6.3 \times 10^{19}\ \mathrm{J})$? (These energy consumption values are from 2006.)

7.8 Work, Energy, and Power in Humans

44. (a) How long can you rapidly climb stairs (116/min) on the 93.0 kcal of energy in a 10.0-g pat of butter? (b) How many flights is this if each flight has 16 stairs?

45. (a) What is the power output in watts and horsepower of a 70.0-kg sprinter who accelerates from rest to 10.0 m/s in 3.00 s? (b) Considering the amount of power generated, do you think a well-trained athlete could do this repetitively for long periods of time?

46. Calculate the power output in watts and horsepower of a shot-putter who takes 1.20 s to accelerate the 7.27-kg shot from rest to 14.0 m/s, while raising it 0.800 m. (Do not include the power produced to accelerate his body.)

Figure 7.42 Shot putter at the Dornoch Highland Gathering in 2007. (credit: John Haslam, Flickr)

47. (a) What is the efficiency of an out-of-condition professor who does $2.10 \times 10^5\ \mathrm{J}$ of useful work while metabolizing 500 kcal of food energy? (b) How many food calories would a well-conditioned athlete metabolize in doing the same work with an efficiency of 20%?

48. Energy that is not utilized for work or heat transfer is converted to the chemical energy of body fat containing about 39 kJ/g. How many grams of fat will you gain if you eat 10,000 kJ (about 2500 kcal) one day and do nothing but sit relaxed for 16.0 h and sleep for the other 8.00 h? Use data from **Table 7.5** for the energy consumption rates of these activities.

49. Using data from **Table 7.5**, calculate the daily energy needs of a person who sleeps for 7.00 h, walks for 2.00 h, attends classes for 4.00 h, cycles for 2.00 h, sits relaxed for 3.00 h, and studies for 6.00 h. (Studying consumes energy at the same rate as sitting in class.)

50. What is the efficiency of a subject on a treadmill who puts out work at the rate of 100 W while consuming oxygen at the rate of 2.00 L/min? (Hint: See **Table 7.5**.)

51. Shoveling snow can be extremely taxing because the arms have such a low efficiency in this activity. Suppose a person shoveling a footpath metabolizes food at the rate of 800 W. (a) What is her useful power output? (b) How long will it take her to lift 3000 kg of snow 1.20 m? (This could be the amount of heavy snow on 20 m of footpath.) (c) How much waste heat transfer in kilojoules will she generate in the process?

52. Very large forces are produced in joints when a person jumps from some height to the ground. (a) Calculate the magnitude of the force produced if an 80.0-kg person jumps from a 0.600–m-high ledge and lands stiffly, compressing joint material 1.50 cm as a result. (Be certain to include the weight of the person.) (b) In practice the knees bend almost involuntarily to help extend the distance over which you stop. Calculate the magnitude of the force produced if the stopping distance is 0.300 m. (c) Compare both forces with the weight of the person.

53. Jogging on hard surfaces with insufficiently padded shoes produces large forces in the feet and legs. (a) Calculate the magnitude of the force needed to stop the downward motion of a jogger's leg, if his leg has a mass of 13.0 kg, a speed of 6.00 m/s, and stops in a distance of 1.50 cm. (Be certain to include the weight of the 75.0-kg jogger's body.) (b) Compare this force with the weight of the jogger.

54. (a) Calculate the energy in kJ used by a 55.0-kg woman who does 50 deep knee bends in which her center of mass is lowered and raised 0.400 m. (She does work in both directions.) You may assume her efficiency is 20%. (b) What is the average power consumption rate in watts if she does this in 3.00 min?

55. Kanellos Kanellopoulos flew 119 km from Crete to Santorini, Greece, on April 23, 1988, in the *Daedalus 88*, an aircraft powered by a bicycle-type drive mechanism (see **Figure 7.43**). His useful power output for the 234-min trip was about 350 W. Using the efficiency for cycling from **Table 7.2**, calculate the food energy in kilojoules he metabolized during the flight.

Figure 7.43 The Daedalus 88 in flight. (credit: NASA photo by Beasley)

56. The swimmer shown in **Figure 7.44** exerts an average horizontal backward force of 80.0 N with his arm during each 1.80 m long stroke. (a) What is his work output in each stroke? (b) Calculate the power output of his arms if he does 120 strokes per minute.

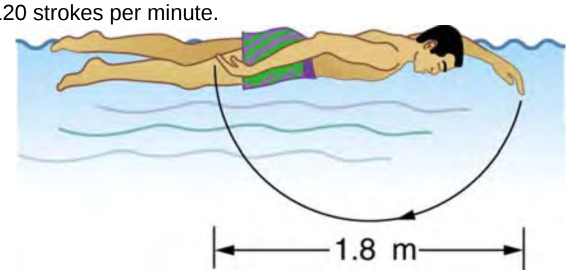

Figure 7.44

57. Mountain climbers carry bottled oxygen when at very high altitudes. (a) Assuming that a mountain climber uses oxygen at twice the rate for climbing 116 stairs per minute (because of low air temperature and winds), calculate how many liters of oxygen a climber would need for 10.0 h of climbing. (These are liters at sea level.) Note that only 40% of the inhaled oxygen is utilized; the rest is exhaled. (b) How much useful work does the climber do if he and his equipment have a mass of 90.0 kg and he gains 1000 m of altitude? (c) What is his efficiency for the 10.0-h climb?

58. The awe-inspiring Great Pyramid of Cheops was built more than 4500 years ago. Its square base, originally 230 m on a side, covered 13.1 acres, and it was 146 m high, with a mass of about $7\times10^9\ \text{kg}$. (The pyramid's dimensions are slightly different today due to quarrying and some sagging.) Historians estimate that 20,000 workers spent 20 years to construct it, working 12-hour days, 330 days per year. (a) Calculate the gravitational potential energy stored in the pyramid, given its center of mass is at one-fourth its height. (b) Only a fraction of the workers lifted blocks; most were involved in support services such as building ramps (see **Figure 7.45**), bringing food and water, and hauling blocks to the site. Calculate the efficiency of the workers who did the lifting, assuming there were 1000 of them and they consumed food energy at the rate of 300 kcal/h. What does your answer imply about how much of their work went into block-lifting, versus how much work went into friction and lifting and lowering their own bodies? (c) Calculate the mass of food that had to be supplied each day, assuming that the average worker required 3600 kcal per day and that their diet was 5% protein, 60% carbohydrate, and 35% fat. (These proportions neglect the mass of bulk and nondigestible materials consumed.)

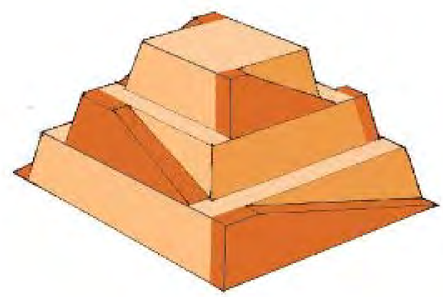

Figure 7.45 Ancient pyramids were probably constructed using ramps as simple machines. (credit: Franck Monnier, Wikimedia Commons)

59. (a) How long can you play tennis on the 800 kJ (about 200 kcal) of energy in a candy bar? (b) Does this seem like a long time? Discuss why exercise is necessary but may not be sufficient to cause a person to lose weight.

7.9 World Energy Use

60. Integrated Concepts

(a) Calculate the force the woman in **Figure 7.46** exerts to do a push-up at constant speed, taking all data to be known to three digits. (b) How much work does she do if her center of mass rises 0.240 m? (c) What is her useful power output if she does 25 push-ups in 1 min? (Should work done lowering her body be included? See the discussion of useful work in **Work, Energy, and Power in Humans**.

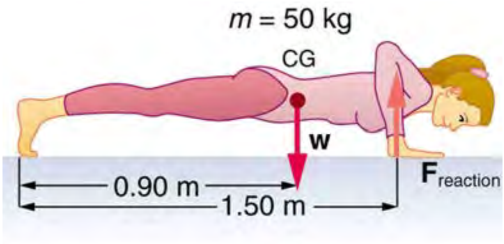

Figure 7.46 Forces involved in doing push-ups. The woman's weight acts as a force exerted downward on her center of gravity (CG).

61. Integrated Concepts

A 75.0-kg cross-country skier is climbing a 3.0° slope at a constant speed of 2.00 m/s and encounters air resistance of 25.0 N. Find his power output for work done against the gravitational force and air resistance. (b) What average force does he exert backward on the snow to accomplish this? (c) If he continues to exert this force and to experience the same air resistance when he reaches a level area, how long will it take him to reach a velocity of 10.0 m/s?

62. Integrated Concepts

The 70.0-kg swimmer in **Figure 7.44** starts a race with an initial velocity of 1.25 m/s and exerts an average force of 80.0 N backward with his arms during each 1.80 m long stroke. (a) What is his initial acceleration if water resistance is 45.0 N? (b) What is the subsequent average resistance force from the water during the 5.00 s it takes him to reach his top velocity of 2.50 m/s? (c) Discuss whether water resistance seems to increase linearly with velocity.

63. Integrated Concepts

A toy gun uses a spring with a force constant of 300 N/m to propel a 10.0-g steel ball. If the spring is compressed 7.00 cm and friction is negligible: (a) How much force is needed to compress the spring? (b) To what maximum height can the ball be shot? (c) At what angles above the horizontal may a child aim to hit a target 3.00 m away at the same height as the gun? (d) What is the gun's maximum range on level ground?

64. Integrated Concepts

(a) What force must be supplied by an elevator cable to produce an acceleration of 0.800 m/s^2 against a 200-N frictional force, if the mass of the loaded elevator is 1500 kg? (b) How much work is done by the cable in lifting the elevator 20.0 m? (c) What is the final speed of the elevator if it starts from rest? (d) How much work went into thermal energy?

65. Unreasonable Results

A car advertisement claims that its 900-kg car accelerated from rest to 30.0 m/s and drove 100 km, gaining 3.00 km in altitude, on 1.0 gal of gasoline. The average force of friction including air resistance was 700 N. Assume all values are known to three significant figures. (a) Calculate the car's efficiency. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

66. Unreasonable Results

Body fat is metabolized, supplying 9.30 kcal/g, when dietary intake is less than needed to fuel metabolism. The manufacturers of an exercise bicycle claim that you can lose 0.500 kg of fat per day by vigorously exercising for 2.00 h per day on their machine. (a) How many kcal are supplied by the metabolization of 0.500 kg of fat? (b) Calculate the kcal/min that you would have to utilize to metabolize fat at the rate of 0.500 kg in 2.00 h. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

67. Construct Your Own Problem

Consider a person climbing and descending stairs. Construct a problem in which you calculate the long-term rate at which stairs can be climbed considering the mass of the person, his ability to generate power with his legs, and the height of a single stair step. Also consider why the same person can descend stairs at a faster rate for a nearly unlimited time in spite of the fact that very similar forces are exerted going down as going up. (This points to a fundamentally different process for descending versus climbing stairs.)

68. Construct Your Own Problem

Consider humans generating electricity by pedaling a device similar to a stationary bicycle. Construct a problem in which you determine the number of people it would take to replace a large electrical generation facility. Among the things to consider are the power output that is reasonable using the legs, rest time, and the need for electricity 24 hours per day. Discuss the practical implications of your results.

69. Integrated Concepts

A 105-kg basketball player crouches down 0.400 m while waiting to jump. After exerting a force on the floor through this 0.400 m, his feet leave the floor and his center of gravity rises 0.950 m above its normal standing erect position. (a) Using energy considerations, calculate his velocity when he leaves the floor. (b) What average force did he exert on the floor? (Do not neglect the force to support his weight as well as that to accelerate him.) (c) What was his power output during the acceleration phase?

Test Prep for AP® Courses

7.1 Work: The Scientific Definition

1. Given **Table 7.7** about how much force does the rocket engine exert on the 3.0-kg payload?

Table 7.7

Distance traveled with rocket engine firing (m)	Payload final velocity (m/s)
500	310
490	300
1020	450
505	312

a. 150 N
b. 300 N
c. 450 N
d. 600 N

2. You have a cart track, a cart, several masses, and a position-sensing pulley. Design an experiment to examine how the force exerted on the cart does work as it moves through a distance.

3. Look at **Figure 7.10**(c). You compress a spring by x, and then release it. Next you compress the spring by $2x$. How much more work did you do the second time than the first?
a. Half as much
b. The same
c. Twice as much
d. Four times as much

4. You have a cart track, two carts, several masses, a position-sensing pulley, and a piece of carpet (a rough surface) that will fit over the track. Design an experiment to examine how the force exerted on the cart does work as the cart moves through a distance.

5. A crane is lifting construction materials from the ground to an elevation of 60 m. Over the first 10 m, the motor linearly increases the force it exerts from 0 to 10 kN. It exerts that constant force for the next 40 m, and then winds down to 0 N again over the last 10 m, as shown in the figure. What is the total work done on the construction materials?

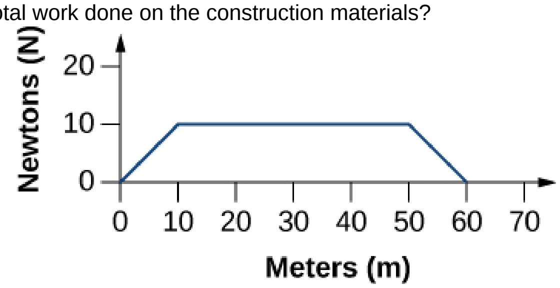

Figure 7.47
a. 500 kJ
b. 600 kJ
c. 300 kJ
d. 18 MJ

7.2 Kinetic Energy and the Work-Energy Theorem

6. A toy car is going around a loop-the-loop. Gravity ____ the kinetic energy on the upward side of the loop, ____ the kinetic energy at the top, and ____ the kinetic energy on the downward side of the loop.
a. increases, decreases, has no effect on
b. decreases, has no effect on, increases
c. increases, has no effect on, decreases
d. decreases, increases, has no effect on

7. A roller coaster is set up with a track in the form of a perfect cosine. Describe and graph what happens to the kinetic energy of a cart as it goes through the first full period of the track.

8. If wind is blowing horizontally toward a car with an angle of 30 degrees from the direction of travel, the kinetic energy will ____. If the wind is blowing at a car at 135 degrees from the direction of travel, the kinetic energy will ____.
a. increase, increase
b. increase, decrease
c. decrease, increase
d. decrease, decrease

9. In what direction relative to the direction of travel can a force act on a car (traveling on level ground), and not change the kinetic energy? Can you give examples of such forces?

10. A 2000-kg airplane is coming in for a landing, with a velocity 5 degrees below the horizontal and a drag force of 40 kN acting directly rearward. Kinetic energy will ____ due to the net force of ____.
a. increase, 20 kN
b. decrease, 40 kN
c. increase, 45 kN
d. decrease, 45 kN

11. You are participating in the Iditarod, and your sled dogs are pulling you across a frozen lake with a force of 1200 N while a 300 N wind is blowing at you at 135 degrees from your direction of travel. What is the net force, and will your kinetic energy increase or decrease?

12. A model drag car is being accelerated along its track from rest by a motor with a force of 75 N, but there is a drag force of 30 N due to the track. What is the kinetic energy after 2 m of travel?
a. 90 J
b. 150 J
c. 210 J
d. 60 J

13. You are launching a 2-kg potato out of a potato cannon. The cannon is 1.5 m long and is aimed 30 degrees above the horizontal. It exerts a 50 N force on the potato. What is the kinetic energy of the potato as it leaves the muzzle of the potato cannon?

14. When the force acting on an object is parallel to the direction of the motion of the center of mass, the mechanical energy ____. When the force acting on an object is antiparallel to the direction of the center of mass, the mechanical energy ____.
a. increases, increases
b. increases, decreases
c. decreases, increases
d. decreases, decreases

15. Describe a system in which the main forces acting are parallel or antiparallel to the center of mass, and justify your answer.

16. A child is pulling two red wagons, with the second one tied to the first by a (non-stretching) rope. Each wagon has a mass of 10 kg. If the child exerts a force of 30 N for 5.0 m, how much has the kinetic energy of the two-wagon system changed?

a. 300 J
b. 150 J
c. 75 J
d. 60 J

17. A child has two red wagons, with the rear one tied to the front by a (non-stretching) rope. If the child pushes on the rear wagon, what happens to the kinetic energy of each of the wagons, and the two-wagon system?

18. Draw a graph of the force parallel to displacement exerted on a stunt motorcycle going through a loop-the-loop versus the distance traveled around the loop. Explain the net change in energy.

7.3 Gravitational Potential Energy

19. A 1.0 kg baseball is flying at 10 m/s. How much kinetic energy does it have? Potential energy?

a. 10 J, 20 J
b. 50 J, 20 J
c. unknown, 50 J
d. 50 J, unknown

20. A 2.0-kg potato has been launched out of a potato cannon at 9.0 m/s. What is the kinetic energy? If you then learn that it is 4.0 m above the ground, what is the total mechanical energy relative to the ground?

a. 78 J, 3 J
b. 160 J, 81 J
c. 81 J, 160 J
d. 81 J, 3 J

21. You have a 120-g yo-yo that you are swinging at 0.9 m/s. How much energy does it have? How high can it get above the lowest point of the swing without your doing any additional work, on Earth? How high could it get on the Moon, where gravity is 1/6 Earth's?

7.4 Conservative Forces and Potential Energy

22. Two 4.0 kg masses are connected to each other by a spring with a force constant of 25 N/m and a rest length of 1.0 m. If the spring has been compressed to 0.80 m in length and the masses are traveling toward each other at 0.50 m/s (each), what is the total energy in the system?

a. 1.0 J
b. 1.5 J
c. 9.0 J
d. 8.0 J

23. A spring with a force constant of 5000 N/m and a rest length of 3.0 m is used in a catapult. When compressed to 1.0 m, it is used to launch a 50 kg rock. However, there is an error in the release mechanism, so the rock gets launched almost straight up. How high does it go, and how fast is it going when it hits the ground?

24. What information do you need to calculate the kinetic energy and potential energy of a spring? Potential energy due to gravity? How many objects do you need information about for each of these cases?

25. You are loading a toy dart gun, which has two settings, the more powerful with the spring compressed twice as far as the lower setting. If it takes 5.0 J of work to compress the dart gun to the lower setting, how much work does it take for the higher setting?

a. 20 J
b. 10 J
c. 2.5 J
d. 40 J

26. Describe a system you use daily with internal potential energy.

27. Old-fashioned pendulum clocks are powered by masses that need to be wound back to the top of the clock about once a week to counteract energy lost due to friction and to the chimes. One particular clock has three masses: 4.0 kg, 4.0 kg, and 6.0 kg. They can drop 1.3 meters. How much energy does the clock use in a week?

a. 51 J
b. 76 J
c. 127 J
d. 178 J

28. A water tower stores not only water, but (at least part of) the energy to move the water. How much? Make reasonable estimates for how much water is in the tower, and other quantities you need.

29. Old-fashioned pocket watches needed to be wound daily so they wouldn't run down and lose time, due to the friction in the internal components. This required a large number of turns of the winding key, but not much force per turn, and it was possible to overwind and break the watch. How was the energy stored?

a. A small mass raised a long distance
b. A large mass raised a short distance
c. A weak spring deformed a long way
d. A strong spring deformed a short way

30. Some of the very first clocks invented in China were powered by water. Describe how you think this was done.

7.5 Nonconservative Forces

31. You are in a room in a basement with a smooth concrete floor (friction force equals 40 N) and a nice rug (friction force equals 55 N) that is 3 m by 4 m. However, you have to push a very heavy box from one corner of the rug to the opposite corner of the rug. Will you do more work against friction going around the floor or across the rug, and how much extra?

a. Across the rug is 275 J extra
b. Around the floor is 5 J extra
c. Across the rug is 5 J extra
d. Around the floor is 280 J extra

32. In the Appalachians, along the interstate, there are ramps of loose gravel for semis that have had their brakes fail to drive into to stop. Design an experiment to measure how effective this would be.

7.6 Conservation of Energy

33. You do 30 J of work to load a toy dart gun. However, the dart is 10 cm long and feels a frictional force of 10 N while going through the dart gun's barrel. What is the kinetic energy of the fired dart?

a. 30 J
b. 29 J
c. 28 J
d. 27 J

34. When an object is lifted by a crane, it begins and ends its motion at rest. The same is true of an object pushed across a rough surface. Explain why this happens. What are the differences between these systems?

35. A child has two red wagons, with the rear one tied to the front by a stretchy rope (a spring). If the child pulls on the front wagon, the ____ increases.

a. kinetic energy of the wagons
b. potential energy stored in the spring
c. both A and B
d. not enough information

36. A child has two red wagons, with the rear one tied to the front by a stretchy rope (a spring). If the child pulls on the front wagon, the energy stored in the system increases. How do the relative amounts of potential and kinetic energy in this system change over time?

37. Which of the following are closed systems?
a. Earth
b. a car
c. a frictionless pendulum
d. a mass on a spring in a vacuum

38. Describe a real-world example of a closed system.

39. A 5.0-kg rock falls off of a 10 m cliff. If air resistance exerts a force of 10 N, what is the kinetic energy when the rock hits the ground?
a. 400 J
b. 12.6 m/s
c. 100 J
d. 500 J

40. Hydroelectricity is generated by storing water behind a dam, and then letting some of it run through generators in the dam to turn them. If the system is the water, what is the environment that is doing work on it? If a dam has water 100 m deep behind it, how much energy was generated if 10,000 kg of water exited the dam at 2.0 m/s?

41. Before railroads were invented, goods often traveled along canals, with mules pulling barges from the bank. If a mule is exerting a 1200 N force for 10 km, and the rope connecting the mule to the barge is at a 20 degree angle from the direction of travel, how much work did the mule do on the barge?
a. 12 MJ
b. 11 MJ
c. 4.1 MJ
d. 6 MJ

42. Describe an instance today in which you did work, by the scientific definition. Then calculate how much work you did in that instance, showing your work.